Exponential Convergence of a Learning Controller for Robot Manipulators

Roberto Horowitz, William Messner, and John B. Moore

Abstract—This note presents the proof for the exponential convergence of a class of learning and repetitive control algorithms for robot manipulators. The learning process involves the identification of the robot inverse dynamics function by having the robot execute a set of tasks repeatedly. Using the concepts of functional persistence of excitation (PE) and functional uniform complete observability (UCO), it is shown that, when a training task is selected for the robot which is persistently excited, the learning controllers are globally exponentially stable. Repetitive controllers are always exponentially stable.

I. INTRODUCTION

In this note we provide exponential convergence proofs for some of the learning and repetitive motion control algorithms introduced in [4]. In [5], the learning control problem was formulated as a functional identification problem. Unknown functions were described in terms of integral equations of the first kind consisting of known kernels and unknown influence functions. The learning process involves the indirect estimation of the unknown functions by estimating the influence functions. In the robot motion control context, robot tasks were associated with tracking a set of desired trajectories and velocities. The objective of the learning algorithm is to identify the robot inverse dynamics function which the control action must generate to achieve perfect tracking. Sufficient conditions for the existence of integral equations for repetitive tasks were presented, and the asymptotic stability of the learning process was demonstrated. The robustness of the learning scheme to noise and unmodeled dynamics was not addressed. However, these properties were apparent from both simulation and experimental results.

In adaptive systems theory, the role of persistence of excitation to achieve exponential stability and robustness in the presence of unmodeled disturbances and dynamics is well understood (cf. [1], [9]). In [5] a linear systems framework was proposed in which concepts such as functional persistence of excitation and functional uniform complete observability were developed to prove the exponential stability of a repetitive learning algorithm presented in [4]; under the assumption that the unknown functions and kernels have a finite eigenfunction expansion. In this note, we will further extend this framework to a class of nonlinear mechanical systems which includes rigid link anthropomorphic robot arms.

II. LEARNING CONTROL

Using the results of Sadegh and Horowitz in [7], it was shown in [4] that the tracking error dynamics of an n degree-of-freedom robot manipulator, under the action of the desired compensation control law (DCCL), satisfy the following differential equation.

\[ \dot{e} = f(t, e) + B(t, e) \left[ \underline{w}(u) - \bar{w}(t, u) \right] \]

(2.1)

where the nonlinear function \( f(t, e) \) is defined in [4]

\[ B(t, e) = \begin{bmatrix} 0 \\ M^{-1}(t, e) \end{bmatrix} \]

(2.2)

and the matrix \( M(\cdot, \cdot) \) is the manipulator generalized inertia matrix. The tracking error state vector is given by

\[ e(t) = [e^T, e^T]^T \]

(2.3)

where \( e(t) \) is the period. The function \( w(\cdot) : A \rightarrow R^n \) represents the unknown desired robot inverse dynamics. The function \( \bar{w}(\cdot) : R^+ \rightarrow A \) is a known function of time, which is associated with the learning task. The compact subspace \( A \) depends on the specified learning tasks. In the general learning control problem, the robot is required to track an arbitrary finite set of desired trajectories

\[ S = \{ x_d \in C^2 | x_d(t) \in E_p, \dot{x}_d(t) \in E_v, \ddot{x}_d(t) \in E_\xi \} \]

(2.4)

where \( E_p, E_v, \) and \( E_\xi \) are compact subsets of \( R^n \). For this case, \( A = E_p \times E_v \times E_\xi \) and

\[ u(t) = [x_d^T, \dot{x}_d^T, \ddot{x}_d^T]^T \]

(2.5)

where \( \bar{w}(u) := M(x_d) \dot{x}_d + c(x_d, \dot{x}_d, \ddot{x}_d) + g(x_d) \)

(2.6)

\( M(\cdot) : R^n \rightarrow R^{n \times n} \) is the generalized inertia matrix, \( c(\cdot, \cdot, \cdot) : R^{3n} \rightarrow R^n \) is the vector due to Coriolis and centripetal accelerations and \( g(\cdot) : R^n \rightarrow R^n \) is the vector due to gravitational torque and forces.

In the repetitive control problem considered in [6], [10], and their references, the robot is required to track a single periodic trajectory \( x_d(t) = x_d(t + T) \in C^2 \), where \( T \) is the period. A related problem is considered in [3], [2]. In this case, \( w(\cdot) \) can be considered as an unknown periodic function of time \( w(\cdot) = \underline{w}(\cdot) + \omega(t \mod T) \) \( A = [0, T] \).

Reference:
The function \( \hat{w}(\cdot, \cdot) : R^+ \times A \rightarrow R^n \) is the estimate of the unknown function \( w(\cdot) \). Defining the function error

\[
\hat{w}(t, u) := w(t) - \hat{w}(t, u) \tag{2.4}
\]
as shown in [7], the error dynamics in (2.1) is \( L_e \) input-output stable, with respect to the pair \( [\hat{w}, e] \), and satisfies Lemma A1 in the Appendix. Furthermore, it is easily shown from results in [7] that \( f(t, e) = F(t, e) e \) where \( F(\cdot, \cdot) \) is a matrix of nonlinear functions, and \( F(t, e) \in L_\infty \) if \( e(t) \in L_\infty \).

We now discuss the general learning rule presented in [4]. The learning control objective consists in the functional adaptation of \( \hat{w}(\cdot, \cdot) \) so that the tracking errors converge to zero. This is done by assuming that the function \( w(u) \) can be represented in the following integral form

\[
w(u) = \int_A K(u, \gamma) c(\gamma) d\gamma \tag{2.5}
\]
where the kernel \( K(\cdot, \cdot) : A \times A \rightarrow R \) is known. The unknown function \( c(\cdot) : A \rightarrow R^n \) must be estimated.

The results of this note are proven under the additional assumption (discussed in [5]) that \( K(\cdot, \cdot) \) has a finite eigenfunction expansion

\[
K(u, \gamma) = \sum_{n=1}^\infty \lambda_n \phi_n(u) \psi_n(\gamma) = \Psi^T(u) \Psi(\gamma) \tag{2.6}
\]
and the \( \phi \) and \( \psi \) are orthonormal over \( A \). This assumption restricts the inverse dynamics function \( w(\cdot) \) on \( \omega(\cdot) \) for the repetitive control cases) to have a finite eigenfunction expansion. However, in practice, this assumption is not too restrictive since \( w(\cdot) \) can be decomposed into \( w(\cdot) = w_0(\cdot) + w_1(\cdot) \), where \( w_0(\cdot) \) has a finite eigenfunction expansion while \( w_1(\cdot) \) contains neglected high-frequency components and can be considered as a disturbance input to the control system. It is not necessary to explicitly determine the orthonormal eigenfunction set \( \{ \cdot \} \). Only a kernel with a finite eigenfunction expansion must be generated.

Selecting a training task with desired trajectory \( u_t(t) : R^+ \rightarrow A \), and defining

\[
w_t(t) = \int_A K(u_t(t), \gamma) c(\gamma) d\gamma \tag{2.7}
\]
the learning rule for generating the function estimate \( \hat{w}(t, u) \) is

\[
\hat{w}(t, u) = \int_A K(u, \gamma) \hat{c}(t, \gamma) d\gamma \tag{2.8}
\]

where \( \hat{c}(\cdot, \cdot) : R \times A \rightarrow R^n \) is the estimate of the influence function \( c(\cdot) \) in (2.5), and is updated by

\[
\frac{\partial}{\partial t} \hat{c}(t, \gamma) = K(u_t(t), \gamma) K_L R^T \hat{e}(t) \tag{2.9}
\]

where \( R = [0 I]^T \) and \( K_L \) is a symmetric positive definite matrix. Defining \( \hat{e}(t, \gamma) = e(\gamma) - \hat{e}(t, \gamma) \) we have

\[
\hat{w}(t, u) = \int_A K(u, \gamma) \hat{c}(t, \gamma) d\gamma \tag{2.10}
\]

\[
\frac{\partial}{\partial t} \hat{c}(t, \gamma) = -K(u_t(t), \gamma) K_L R^T \hat{e}(t). \tag{2.11}
\]

A. Persistence of Excitation

In this section, we define the concept of persistence of excitation within the repetitive control framework. Persistence of excitation plays an important role in the exponential stability proof that will be presented in Section III.

**Definition 2.1 – Persistently Exciting (PE) Kernel:** A training task has a PE desired trajectory \( u_t(t) : R^+ \rightarrow A \), or correspondingly, the kernel \( K(u_t(t), \gamma) \) is PE on \( A \) when, for some scalars \( \alpha_1, \alpha_2, \gamma > 0 \) and for all influence functions \( c(\cdot) : A \rightarrow R^n \)

\[
\alpha_1 \int_A |c(\gamma)|^2 d\gamma \geq \frac{1}{\gamma^2} \int_0^\gamma |w_t(\gamma)|^2 d\gamma \tag{2.12}
\]

\[
\geq \alpha_1 \int_A |c(\gamma)|^2 d\gamma \tag{2.13}
\]

for all \( t \). \( w_t(\cdot) \) is defined in (2.7).

Reference [5] presents a technique for testing of a given desired trajectory which will yield a PE kernel.

B. Repetitive Learning Control

In repetitive control applications, the repetitive inverse dynamics function \( w_t(\cdot) \) can be estimated by assuming that it can be represented in the following integral form

\[
w_t(t) = \int_0^T K(t, \tau) c(\tau) d\tau \tag{2.14}
\]

where \( K(t, \tau) : R_r \times [0, T] \rightarrow R \) is a kernel which satisfies \( K(t + T, \tau) = K(t, \tau) \), and \( c(\cdot) : [0, T] \rightarrow R^n \) is the influence function.

The repetitive learning rule for generating the function estimate \( w_t(\cdot) \) is

\[
\hat{w}_t(t) = \int_0^T K(t, \tau) \hat{c}(t, \tau) d\tau \tag{2.15}
\]

\[
\frac{\partial}{\partial t} \hat{c}(t, \tau) = K(t, \tau) K_L \hat{e}(t) \tag{2.16}
\]

where \( \hat{c}(t, \cdot) : R_r \times [0, T] \rightarrow R^n \) is the influence function estimate.

**Lemma 2.1:** If the repetitive inverse dynamics function \( w_t(\cdot) \) satisfies (2.14), where the kernel \( K(t, \tau) \) has a finite eigenfunction expansion as in (2.6) (with \( u = t \) and \( A = [0, t] \)), then the kernel \( \hat{c}(t, \cdot) \) is PE.

**Proof:** By direct calculations it follows that

\[
\lambda_{\max} \int_0^T |c(\tau)|^2 d\tau \geq \int_0^T |w_t(\tau)|^2 d\tau \tag{2.17}
\]

\[
\geq \lambda_{\max} \int_0^T |c(\tau)|^2 d\tau \tag{2.18}
\]

for all \( t \), where \( \lambda_{\max} \geq \lambda_i \geq \lambda_{\min} \) for \( i = 1 \cdots N \).

**Remark:** It is reasonable to assume that \( w_t(\cdot) \) can be approximated by a function which satisfies (2.14), where the kernel \( K(t, \tau) \) is known. For example, for any \( w_t(\cdot) \) which can be approximated by a finite Fourier series expansion, \( K(t, \tau) \) can be defined as

\[
K(t, \tau) = c_0 + \sum_{n=1}^{N} \left[ c_n \cos \frac{2 \pi}{T} - \cos \frac{2 \pi}{T} \right. \\
+ d_n \sin \frac{2 \pi}{T} \sin \frac{2 \pi}{T} \right. \tag{2.19}
\]

\( c_n, d_n \in 0 \forall n \leq N \). There is a large class of kernels which can be described by (2.19).

We are now ready to state the main result of the note.

III. EXponential Stability

**Theorem 3.1:** For the manipulator error dynamics given by (2.1), and the learning rule given by (2.8)-(2.9). Under the following conditions: 1) \( w(\cdot) \) can be expressed in the integral form (2.5), with a known kernel \( K(u, \gamma) \), which has a finite eigenfunction
expansion. 2) $K(u(t), \gamma)$ satisfies the following integral bounds:

$$\sup_{u(t) \in A} \int_{\gamma}^1 K(u(t), \gamma) \, d\gamma = \epsilon < \infty$$  

(3.20)

$$\sup_{u(t) \in A} \int_{\gamma}^1 K(u(t), \gamma) \, d\gamma = \epsilon_d < \infty$$  

(3.21)

where $K(u(t), \gamma) = [\delta K(u(t), \gamma)/\delta \dot{u}] \dot{u}$. 3) A training task with corresponding desired trajectory $u_T(t)$ is chosen such that the kernel $K(u_T(t), \gamma)$ is PE, then the system is exponentially stable.

**Proof:** We shall first prove, as shown in [4], that $\epsilon(t) \in L_2 \cap L_{\infty}$ and $\tilde{w}(t) \in L_{\infty}$. Define the Lyapunov functional candidate

$$V(t, e, \tilde{e}) = V_1(t, e) = \frac{1}{2} \int_{A} \tilde{e}(t, \gamma)^T K(t, \gamma) \tilde{e}(t, \gamma) \, d\gamma$$  

(3.22)

where $V_1$, is the function in Lemma A1 in the Appendix. Differentiating $V$ with respect to time, along the error trajectories in (2.1) and utilizing the learning rule given by (2.8)-(2.9), we obtain

$$V = -\alpha V_1(t, e) + e^T R \tilde{w} + \int_{A} e^T K(t, \gamma) \tilde{e}(t, \gamma) \, d\gamma$$

$$= -\alpha V_1(t, e) + e^T R \tilde{w} - e^T R \tilde{w}$$

$$\leq -\lambda \epsilon(t), \lambda > 0,$$  

(3.23)

which in turn, results in $\epsilon(t) \in L_2 \cap L_{\infty}$ and $\tilde{w}(t) \in L_{\infty}$.

We now show that the system given by the error dynamics (2.1) and the learning rule (2.8)-(2.9) satisfies the following property, under the assumptions stated in the theorem.

**Property 3.1:** There exists $\beta_1, \beta_2, T > 0$ such that, for all $t$

$$\beta_1 \epsilon(t)^T \epsilon(t) + \int_{\gamma}^1 \tilde{e}(t, \gamma)^T \tilde{e}(t, \gamma) \, d\gamma$$

$$\leq \beta_2 \epsilon(t)^T \epsilon(t) + \int_{A} \tilde{e}(t, \gamma)^T \tilde{e}(t, \gamma) \, d\gamma,$$  

(3.24)

**Remark:** For linear systems, Property 3.1 is equivalent to the system being uniformly completely observable (UCO) with respect to $e$. However, the manipulator error dynamics in Section II-B are nonlinear.

Utilizing the fact that $f(t, e) = F(t, e) e$, the manipulator error dynamics can be represented by the block diagram in Fig. 1. This block diagram is in turn equivalent to the block diagram in Fig. 2. Notice that $M^{-1}(t, 0)$ and $F(t, 0)$ are time varying matrices which do not depend on the state.

Referring to the symbols defined in Fig. 2, $e \in L_2 \cap L_{\infty}$ implies that $\|F(t, e)\| \in L_2$ and that $\tilde{w}(t) \in L_2$. This is due to the fact that the product of a $L_2$ signal with a $L_2$ signal is a $L_2$ signal. Since $\tilde{w}(t) \in L_2$ and, by Lemma A4 in the Appendix, $\|M^{-1}(t, e) - M^{-1}(t, 0)\| \in L_2$, we also have that $u_T(t) \in L_2$. Thus, $u(t) = u_T(t) + w_T(t) \in L_2$.

Since uniform complete observability is not affected by the injection of $L_2$ signals [9], the system in Fig. 2 will satisfy (3.24) if the following linear time-varying system is UCO.

$$\frac{d}{dt} \begin{bmatrix} \tilde{e}_p \\ \tilde{e}_s \end{bmatrix} = \begin{bmatrix} F(t, 0) \\ M(t, 0) \end{bmatrix} \tilde{w}(t)$$  

(3.25)

$$\tilde{e}(t) = \int_{\gamma} K(u_T(t), \gamma) \tilde{e}(t, \gamma) \, d\gamma$$

$$\frac{d}{dt} \tilde{e}(t, \gamma) = -K(u_T(t), \gamma) K_R \tilde{e}(t, \gamma).$$

By defining the matrix kernels

$$C(t, \gamma) = \begin{bmatrix} 0 & M^{-1}(t, 0) \end{bmatrix} K(u_T(t), \gamma)$$  

(3.26)

$$J(t, \gamma) = -K(u_T(t), \gamma) K_R R^T$$  

(3.27)

we arrive at a system of the form of Lemma A5 in the Appendix. Thus, by Lemma A5, to show the uniform complete observability of the system in (3.25), it is only necessary to show that the kernel $C(t, \gamma)$ is PE.

For an arbitrary influence function $e(t) \in A$, with a finite eigenfunction expansion, we define the vector

$$z(t) = \int_{A} C(t, \gamma) e(\gamma) \, d\gamma.$$  

(3.28)

By (3.26) we obtain

$$\lambda_m \leq |w_T(t)|^2 \leq |z(t)|^2 \leq \lambda_M |w_T(t)|^2$$  

(3.29)

where $\lambda_m$ and $\lambda_M$ are, respectively, the smallest and largest eigenvalues of $M(t, 0)$, and $w_T(t)$ is defined in (2.7). Thus, from Definition 2.1, the kernel $C(t, \gamma)$ is PE if the kernel $K(u_T(t), \gamma)$ is PE. Consequently, the linear time varying system in (3.25) is UCO and the system given by error dynamics (2.1) and the learning rule (2.8)-(2.9) satisfies Property 3.1 under the condition that $K(u_T(t), \gamma)$ is PE. By (3.23) and (3.24), there exists an integer
I > 0 and some $\beta > 0$ such that
\[
\int_t^{t+\tau} \dot{V} \, dt \leq -\beta \int_t^{t+\tau} |e(t)|^2 \, dt
\]
\[
\leq -\beta \left[ \frac{1}{2} |e(t)|^2 + \frac{1}{2} \int_A \dot{\epsilon}(t, \gamma) \dot{\epsilon}(t) \, d\gamma \right]
\]
\[
= -\beta' V(t, e, \dot{e}) \quad \beta' > 0. \quad \text{(3.30)}
\]
This implies that the system decays exponentially to zero. Q.E.D.

Theorem 3.2: Consider the system described by the error dynamics in (2.1) and the adaptation law in (2.15) and (2.16). Assume that the robot is expected to execute a training task with a periodic desired trajectory $u_d(t) = u_r(t + T)$. If in addition: 1) the repetitive inverse dynamic function $\hat{\mathbf{N}}(t)$ satisfies (2.14) with a known periodic kernel $K(t, \tau)$ which has a finite eigenfunction expansion. 2) $K(t, \tau)$ and $K(t, r)$ satisfy the integral bounds (3.20) and (3.21), respectively, (with $A = [0, T]$ and $u = r$), then the system is globally exponentially stable.

Proof: Theorem 3.2 follows directly from Lemma 2.1 and Theorem 3.1.

IV. Conclusion

The exponential convergence of some of the learning and repetitive control algorithms presented in [4] has been demonstrated under the additional assumptions that the kernels in the learning integral equations have a finite eigenfunction expansion and are PE. This result strengthens the asymptotic convergence results derived in [4] and corroborates the experimentally observed robustness of the repetitive learning control algorithms in [4]. The exponential convergence of the delayed learning algorithms presented in [4] is in progress as well as exponential stability results for learning rules using least square kernels with forgetting factors. As a final remark we point out that the desired compensation adaptive law (DCAL) in [7], under persistence of excitation, is also exponentially convergent. The proof of this result is very similar to the one presented in this note.

APPENDIX

Lemma A1 [7]: For the manipulator dynamics given by (2.1), consider the Lyapunov function candidate defined by
\[
V(t, e) = \frac{1}{2} \dot{e}^T \left[ F_e + \lambda t I \right] e \quad \text{with} \quad \lambda > 0, \quad F_e = \text{the position error feedback gain of the DCCL, and} \quad \lambda = \text{the constant used in (2.2).}
\]
where $\lambda$ is the average of the maximum and minimum eigenvalues of the generalized inertia matrix over the desired trajectory, $F_e$ is the position error feedback gain of the DCCL, and $\lambda$ is the constant used in (2.2). This Lyapunov function has the following property.
\[
\frac{d}{dt} V(t) = -\alpha V + e^T R \ddot{e}
\]
where $R = [0 \ I]^T$ and $\alpha > 0$.

Proof: See [7]. (See [4] to translate the notation of (2.1) to the notation of [7].)

Lemma A2: Let $M(t, e) = M_x(x, e) = M_x(x)$ be the generalized inertia matrix. If $e \in L_2 \cap L_\infty$ then
\[
\int_0^\infty \| M(t, e) \|^2 \, dt < \infty.
\]
Proof: $M(t, e) \in C^\infty$. Defining $M_x(x) = [m_x(x)]$, then $\partial m_x(x, e)/\partial e \in L_\infty$. From the mean value theorem, we have
\[
\| M(t, e) - M(t, 0) \|^2 = \frac{\partial m_x(t, e)}{\partial e} |(u, x) \| \frac{\partial m_x(t, e)}{\partial e} |(u, x) e |^2 \quad \text{(A.4)}
\]
where $e = \lambda x$, $\lambda \in [0, 1]$. Since $e \in L_2$, the result follows from the fact that the product of an $L_2$ signal with a $L_\infty$ signal is a $L_2$ signal.

Lemma A3: Let $A(t), B(t)$ be Hermitian matrices such that $A(t), B(t) > \alpha I$. If $e^T(t) = A(t) - B(t) \|^2 \, dt < \infty$ then
\[
\int_0^\infty \| A(t)^{-1} - B(t)^{-1} \|^2 \, dt < \infty. \quad \text{(A.5)}
\]

Proof: Utilizing the equality
\[
A^{-1} - B^{-1} = (A^{-1} + B^{-1})(B - A)(A + B)^{-1}
\]
we can write
\[
\| A(t)^{-1} - B(t)^{-1} \| \leq \| A(t) + B(t) \| \cdot \| A(t)^{-1} + B(t)^{-1} \| \cdot \| A(t) - B(t) \|
\]
\[
< \frac{1}{2} \alpha \| A(t) - B(t) \|
\]
\[
= \frac{1}{\alpha^2} \| A(t) - B(t) \|. \quad \text{(A.7)}
\]
Thus, we have that
\[
\int_0^\infty \| A(t)^{-1} - B(t)^{-1} \|^2 \, dt < \frac{1}{\alpha^2} \int_0^\infty \| A(t) - B(t) \|^2 \, dt < \infty. \quad \text{(A.8)}
\]

Lemma A4: If $e \in L_2 \cap L_\infty$ then
\[
\int_0^\infty \| M^{-1}(t, 0) - M^{-1}(t, e) \|^2 \, dt < \infty. \quad \text{(A.9)}
\]

Proof: The result follows immediately from Lemmas A2 and A3.

Lemma A5: Consider the system
\[
\dot{x}_i(t, \gamma) = J(t, \gamma) x_i(t), \quad x_i(t_0, \gamma) \quad \text{on} \ A
\]
\[
\dot{y}(t) = \xi(t, \gamma)
\]
\[
\xi(t) = \xi(t).
\]
Consider also that $J(t, \gamma)$ is PE, $C(t, \gamma)$, $\dot{C}(t, \gamma)$ and $\dot{J}(t, \gamma)$ are integrally bounded as in (3.20). Then for an arbitrary $x_i(t_0, \gamma)$ on $A$, the system (A.10) is UCO, with respect to the output $y$.

Proof: The proof is exactly the same as the proof of [5]. Lemma 4.4, with the constant matrix $A$ in [5, Lemma 4.4] replaced with a bounded time-varying matrix $A(t)$.

REFERENCES


---

On Output Deadbeat Control of Discrete-Time Multivariable Systems

S. K. Spurgeon and A. C. Pugh

Abstract—A direct method to compute output deadbeat controls for linear multivariable systems using stable numerical methods has recently been presented in the literature. This method utilizes the conventional optimal control problem with no cost on the controls and a state weighting matrix to ensure cancellation of all stable transmission zeros. This note outlines an extension to the algorithm which will be shown to provide similar theoretical insights when dealing with the case of a system with multiple stable transmission zeros. Some misprints in the numerical example used to illustrate the algorithm in the original paper are corrected. The corrected example then turns out to justify the necessity of the proposed extension.

I. INTRODUCTION

A novel numerical method for computing output deadbeat controls for linear multivariable systems has been proposed in this TRANSATIONS by Marrari, Emami-Naeini, and Franklin [1]. The method considers the following linear time-invariant discrete-time representation

\[ x(i + 1) = \Phi x(i) + \Gamma u(i) \]

\[ y(i) = Hx(i) \]

where \( x \in \mathbb{R}^m \), \( u \in \mathbb{R}^n \), \( y \in \mathbb{R}^r \) \((r \geq m)\) and \( \Phi, \Gamma, \) and \( H \) are constant, compatibility dimensioned matrices with \((\Phi, \Gamma)\) reachable and rank \((\Gamma) = m\). It is assumed that the system has \( p \) stable transmission zeros. A feedback law of the form

\[ u(i) = -Kx(i) \]

is determined such that

\[ y(i) = Hx(i) = H(\Phi - \Gamma K)x_0 = 0 \]

where \( \mu \) is the minimum number of time steps required to reach the origin from an arbitrary initial state \( x_0 \). This is achieved by choosing \( K \) such that \( n - \mu \) eigenvalues of \( \Phi \) are at the origin and the remaining \( p \) nonzero eigenvalue/eigenvector pairs are assigned to coincide with the \( p \) stable transmission zeros \( z_k \) (assumed for the moment to be distinct) and the corresponding state zero directions, \( v_k \), defined by

\[ \begin{bmatrix} z_k - \Gamma \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}, \quad v_k 

(5)

This ensures that the maximum number of stable modes is made unobservable as is required for a minimum-time output deadbeat controller.

Output Deadbeat Control Algorithm

The output deadbeat control algorithm [1] is formulated as follows:

1) Form

\[ T = \begin{bmatrix} T_1 & T_2 \end{bmatrix} \]

where

\[ T_1 = \begin{bmatrix} v_1, v_2, \ldots, v_p \end{bmatrix} \]

and

\[ T_2 = \begin{bmatrix} s_1, s_2, \ldots, s_p \end{bmatrix} \]

where the \( s_i \), \( i = p + 1, \ldots, n \) are chosen such that \( \{v_1, \ldots, v_p, s_1, \ldots, s_p\} \) forms a basis for \( \mathbb{R}^n \).

2) Transform the system (1), (2):

\[ \Phi_T = T^{-1}\Phi \quad \Gamma_T = T^{-1}\Gamma \quad H_T = HT \]

where

\[ \Phi_T = \begin{bmatrix} \Phi_{T11} & \Phi_{T12} \\ \Phi_{T21} & \Phi_{T22} \end{bmatrix} \]

\[ \Gamma_T = \begin{bmatrix} \Gamma_{T1} \\ \Gamma_{T2} \end{bmatrix} \]

\[ H_T = \begin{bmatrix} 0 & H_{T2} \end{bmatrix} \]

\[ p \quad n - p \]

and define a compatibly partitioned stabilizing state feedback control law (3)

\[ K_T = \begin{bmatrix} K_{T11} & K_{T12} \end{bmatrix} \]

3) Choose \( H_{T2} \) so that \( (\Phi_{T22}, \Gamma_{T22}, H_{T2}) \) has no finite transmission zeros and let

\[ H_T = \begin{bmatrix} 0 & H_{T2} \end{bmatrix} \]

4) Solve the optimal control problem for \((\Phi, \Gamma)\) with a state