This project gives you a chance to review your programming skills. It also presents a simple model of the motion of a group of point vortices. Recall from your fluid mechanics course that a point vortex placed at the origin has the induced velocity field

$$\mathbf{u}(\mathbf{r}) = (u_r, u_\theta) = \left(0, \frac{\Gamma}{2\pi r}\right)$$

(1)

where we used a cylindrical coordinate system \((r, \theta, z)\) and \(\Gamma\) is the strength of the vortex. We rewrite this expression when a vortex of strength \(\Gamma_j\) is placed at an arbitrary location \(\mathbf{r}_j\)

$$\mathbf{u}_j(\mathbf{r}) = \frac{\Gamma_j}{2\pi r_j^2} \mathbf{k} \times (\mathbf{r} - \mathbf{r}_j) = \frac{\Gamma_j}{2\pi \left[ (x-x_j)^2 + (y-y_j)^2 \right]} \left[ -(y-y_j), (x-x_j) \right]$$

(2)

where \(r_j = |\mathbf{r} - \mathbf{r}_j|\) and \(\mathbf{k}\) is the unit vector in the \(z\)-direction. Now, consider the motion of a group of \(n\) vortices \(\Gamma_i\), where we use the subscript \((i)\) to identify vortices. A vortex moves with the total induced velocity due to the rest of the vortices in the group

$$\mathbf{U}_i = \sum_{j=1, j\neq i}^{n} \mathbf{u}_j.$$  

(3)

In explicit form

$$\mathbf{U}_i = \left[ \sum_{j=1, j\neq i}^{n} \frac{\Gamma_j}{2\pi \left[ (x_i-x_j)^2 + (y_i-y_j)^2 \right]} \left[ -(y_i-y_j), (x_i-x_j) \right] \right].$$

(4)

Following Helmholtz’ law, the vortex follows the fluid

$$\frac{d\mathbf{r}_i}{dt} = \mathbf{U}_i(\mathbf{r}_i - \mathbf{r}_j).$$

(5)

Let us consider \(n\) vortices \(\Gamma_j\) placed at arbitrary locations \(\mathbf{r}_j\). Let us approximate equation (5) as

$$\Delta \mathbf{r}_i \approx \mathbf{U}_i \Delta t.$$  

(6)
Write a code in a language of your choice that calculates the position of each vortex as a function of the time

\[ \mathbf{r}_i(t + \Delta t) = \mathbf{r}_i(t) + \Delta \mathbf{r}_i. \]  

(7)

Alternatively, you can use an integration scheme of your choice to integrate the simultaneous ordinary differential equations (5).

**Test Cases**

As test cases to verify your code, confirm that:

(a) For \( n = 2 \) and \( \Gamma_1 = -\Gamma_2 \), the vortex pair moves at a constant velocity.

(b) For \( n = 2 \) and \( \Gamma_1 = \Gamma_2 \), the vortex pair orbits at a constant angular velocity around their centroid.

(c) \( n \) identical vortices placed on a circle at equal intervals orbit at a constant angular velocity around the center of the circle. Try \( n = 3, 5, 10 \).

(d) A vortex placed in a corner moves on a trajectory \( r \sin 2\theta = \text{const} \).

**Calculations**

Use your code to calculate the trajectories of the following symmetric vortex systems whose initial configurations \((x, y, \Gamma)_i\) are given as:

1. Symmetric co-rotating vortex pairs: Vortex wake of a wing with inboard flaps deployed

   \[ \left[ (-b/2, 0, -\Gamma), (-b/4, 0, -\gamma\Gamma), (b/4, 0, +\gamma\Gamma), (b/2, 0, +\Gamma) \right] \]  

   (8)

2. Symmetric counter-rotating vortex pairs: Vortex wake of a wing with tip flaps deployed

   \[ \left[ (-b/2, 0, -\Gamma), (-b/4, 0, +\gamma\Gamma), (b/4, 0, -\gamma\Gamma), (b/2, 0, +\Gamma) \right] \]  

   (9)

3. Finally, a new twist, a symmetric co-rotating vortex pairs, a four-vortex system over ground at \( y = 0 \), simulating a aircraft during takeoff or landing.

   \[ \left[ (-b/2, b/2, -\Gamma), (-b/4, b/2, -\gamma\Gamma), (b/4, b/2, +\gamma\Gamma), (b/2, b/2, +\Gamma) \right] \]  

   (10)

where \( b \) and \( \Gamma \) are positive constants. Take the values of the parameter \( \gamma = 0.2, 0.4, 0.6 \). Carry out your calculations for long enough times to observe the cyclic nature of the vortex motion. In the last configuration, you must consider the image vortex system located at \( y = -b/2 \) to satisfy the impermeability condition at the ground plane, hence, it is an eight-vortex system!
4th Order Runga-Kutta Scheme

\[ y' = f(x, y) \]

\[ y_{n+1} = y_n + \frac{1}{6} k_1 + \frac{1}{3} k_2 + \frac{1}{3} k_3 + \frac{1}{6} k_4 + O(h^5) \]

\[ k_1 = hf(x_n, y_n) \]
\[ k_2 = hf(x_n + \frac{1}{2} h, y_n + \frac{1}{2} k_1) \]
\[ k_3 = hf(x_n + \frac{1}{2} h, y_n + \frac{1}{2} k_2) \]
\[ k_4 = hf(x_n + h, y_n + k_3) \]

Report

Include in your report an abstract, a description of the problem, formulation, and implementation of your code. Then, discuss the flow pattern in each case. The figures must be clearly labeled. Also, be prepared to submit an electronic copy of your code if asked.