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CHAPTER

TWO

VARIATIONAL FORMULATION AND APPROXIMATION

2.1 SOME ANCILLARY CONCEPTS AND FORMULAS

2.1.1 Introduction

The finite-element method is a piecewise application of a variational method. Therefore, we begin with a study of the variational methods. There are two basic steps in the variational solution of differential equations:

1. To cast a given differential equation in variational form
2. To determine the approximate solution using a variational method, such as the Ritz method, the Galerkin method, or other methods

The term "variational formulation" is used in the present study to mean the *weak* formulation in which a given differential equation is recast in an equivalent integral form by trading the differentiation between a test function and the dependent variable. For most linear problems the weak formulation is equivalent to the minimization of a quadratic functional $I(u)$, called the *total potential energy* in solid mechanics problems. Analogous to the necessary condition for the minimum of an ordinary function, the necessary condition for a quadratic functional is that its first derivative (or first variation) with respect to the dependent variable be zero. From calculus of variations one knows that the minimizing function is the true solution of the differential equation. This fact provides us the motivation to study the variational formulation of a given differential equation.

In a variational method the dependent variable of a given problem is approximated by a linear combination of appropriately chosen functions: $u = \Sigma c_j \phi_j$. The parameters c_j are determined such that the function u minimizes the functional $I(u)$ (or, u satisfies the weak formulation of the problem).

The variational form of a given differential equation has several interesting features that facilitate the approximate solution. We illustrate them via an example.

Consider the problem of finding the solution w to the differential equation

$$\frac{d^2}{dx^2}\left[b(x)\,\frac{d^2w}{dx^2}\right] + f(x) = 0 \qquad \text{for } 0 < x < L \tag{2.1}$$

subject to the end conditions (or boundary conditions)

$$w(0) = \frac{dw}{dx}(0) = 0 \qquad \left(b\,\frac{d^2w}{dx^2}\right)\bigg|_{x=L} = M_0 \qquad \left[\frac{d}{dx}\left(b\frac{d^2w}{dx^2}\right)\right]_{x=L} = 0 \tag{2.2}$$

This equation arises, for example, in the study of the elastic bending of beams. In this case w denotes the transverse deflection of the beam, L is the total length of the beam, $b(x) > 0$ is the flexural rigidity (i.e., the product of modulus of elasticity and moment of inertia) of the beam, $f(x)$ is the transverse distributed load, and M_0 is the bending moment (see Fig. 2.1*a*). The solution w is called the *dependent variable* of the problem, and all other quantities (L, b, f, M_0) which are known in advance are called the *data* of the problem.

When $b(x)$ and $f(x)$ are continuous functions of x in $(0, L)$, the data are said to be *smooth*, for which case the solution w to the problem exists and satisfies the differential equation (2.1) at every point x in $(0, L)$ as well as the boundary conditions (2.2) at the boundary points. As an example consider the case in which b and f are nonzero constants. Then the exact solution of Eqs. (2.1) and (2.2) is given by

$$w(x) = \frac{2M_0 - fL^2}{4b}x^2 + \frac{fL}{6b}x^3 - \frac{f}{24b}x^4 \tag{2.3}$$

Thus the solution w and its derivatives up to fourth order are well defined (i.e., they exist and are single-valued) at every point of the domain $(0, L)$.

In most practical situations the data given in a problem are not smooth (i.e., not continuous everywhere in the domain). For example, the flexural rigidity may be discontinuous (e.g., in the case of a composite beam made of dissimilar materials or in the case of a stepped beam), or the transverse loading f may be discontinuous. Suppose that (see Fig. 2.1*b*)

$$f(x) = f_0 H(a - x) \tag{2.4}$$

and $b(x)$ is continuous. Here $H(a - x)$ denotes the *Heaviside step function*,

$$H(a - x) = \begin{cases} 1 & \text{for } x < a \\ 0 & \text{for } x > a_0 \end{cases} \tag{2.5}$$

In this case, the fourth derivative of the solution (i.e., w) does not exist (i.e., is not single-valued) at $x = a$. Therefore, the exact solution w to Eqs. (2.1) and (2.2) does not exist in the classical sense [i.e., w must satisfy the differential equation (2.1) at all points of the domain]. Similar difficulties are encountered when w and/or its derivatives are specified at points between the endpoints $x = 0$ and $x = L$.

Multiplying Eq. (2.1) with a function v, called the *test function*, that is twice differentiable and satisfies the conditions

$$v(0) = \frac{dv}{dx}(0) = 0$$

integrating the first term twice by parts, and using the boundary conditions (2.2),

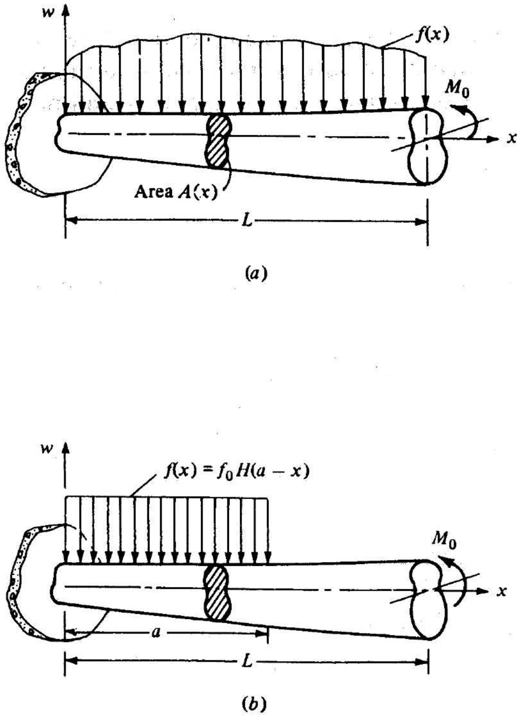

Figure 2.1 Problems with continuous and discontinuous data (H is the Heaviside step function). (*a*) A cantilevered beam with continuous load. (*b*) A cantilevered beam with discontinuous load.