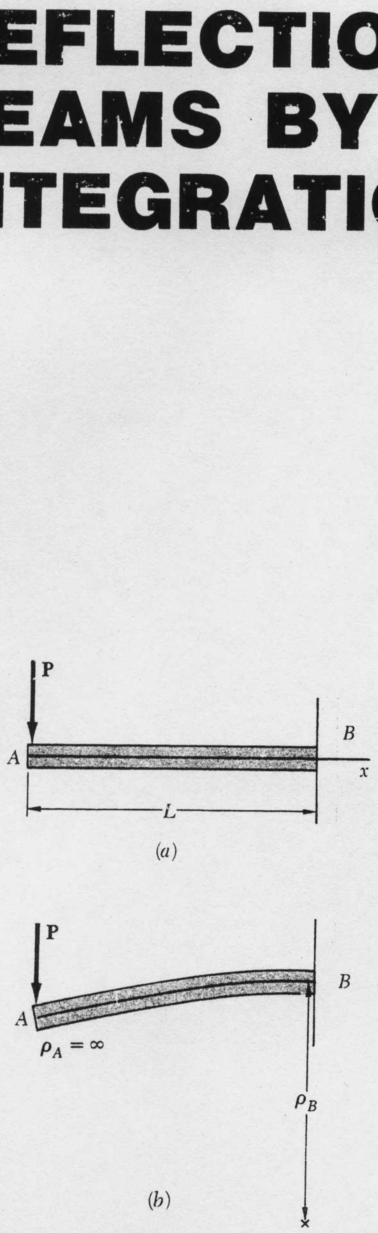
Chapter 8
DEFLECTION OF BEAMS BY INTEGRATION



(a)

(b)

Fig. 8.1

8.1. INTRODUCTION

We saw in Sec. 4.4 that a prismatic beam subjected to pure bending is bent into an arc of circle and that, within the elastic range, the curvature of the neutral surface may be expressed as

$$\frac{1}{\rho} = \frac{M}{EI} \tag{4.21}$$

where M is the bending moment, E the modulus of elasticity, and I the moment of inertia of the cross section about its neutral axis.

When a beam is subjected to a transverse loading, Eq. (4.21) remains valid for any given transverse section, provided that Saint-Venant's principle applies. However, both the bending moment and the curvature of the neutral surface will vary from section to section. Denoting by x the distance of the section from the left end of the beam, we write

$$\frac{1}{\rho} = \frac{M(x)}{EI} \tag{8.1}$$

Consider, for example, a cantilever beam AB of length L subjected to a concentrated load **P** at its free end A (Fig. 8.1*a*). We have $M(x) = -Px$ and, substituting into (8.1),

$$\frac{1}{\rho} = -\frac{Px}{EI}$$

which shows that the curvature of the neutral surface varies linearly with x, from zero at A, where ρ_A itself is infinite, to $-PL/EI$ at B, where $|\rho_B| = EI/PL$ (Fig. 8.1*b*).

Consider now the overhanging beam AD of Fig. 8.2, which supports two concentrated loads as shown. From the free-body diagram of the beam (Fig. 8.3*a*), we find that the reactions at the supports are $R_A = 1$ kN and $R_C = 5$ kN, respectively, and draw the corresponding bending-moment diagram (Fig. 8.3*b*). We note from the diagram that M, and thus the curvature of the beam, are zero at both ends of the beam, and also at a point E located at $x = 4$ m. Between A and E the bending moment is positive and the beam is concave upward; between E and D the bending moment is negative and the beam is concave downward (Fig. 8.3*c*). We also note that the largest value of the curvature (i.e., the smallest value of the radius of curvature) occurs at the support C, where $|M|$ is maximum.

From the information obtained on its curvature, we may get a fairly good idea of the shape of the deformed beam. However, the analysis and design of a beam usually require more precise information on the *deflection* and the *slope* of the beam at various points. Of particular importance is the knowledge of the *maximum deflection* of the beam. In this chapter we shall use Eq. (8.1) to obtain a relation between the deflection y measured at a given point Q on the axis of the beam and the distance x of that point from some fixed origin (Fig. 8.4). The relation obtained is the equation of the *elastic curve*, i.e., the equation of the curve into which the axis of the beam is transformed under the given loading (Fig. 8.4*b*).†

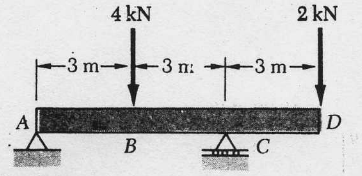


Fig. 8.2

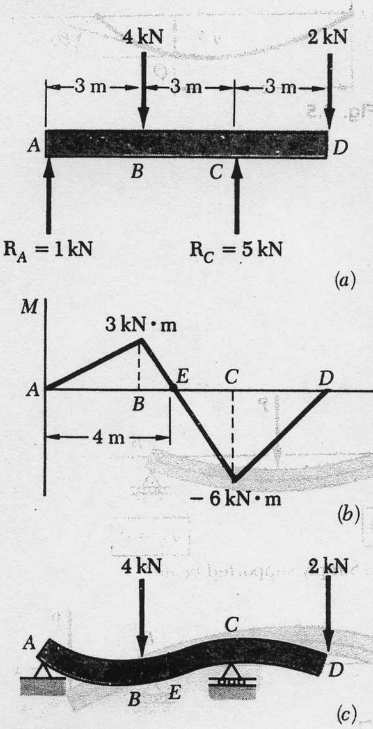


Fig. 8.3

8.2. EQUATION OF THE ELASTIC CURVE

We first recall from elementary calculus that the curvature of a plane curve at a point $Q(x, y)$ of the curve may be expressed as

$$\frac{1}{\rho} = \frac{\dfrac{d^2y}{dx^2}}{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}} \tag{8.2}$$

where dy/dx and d^2y/dx^2 are the first and second derivatives of the function $y(x)$ represented by that curve. But, in the case of the elastic curve of a beam, the slope dy/dx is very small, and its square is negligible compared to unity. We may write, therefore,

$$\frac{1}{\rho} = \frac{d^2y}{dx^2} \tag{8.3}$$

Substituting for $1/\rho$ from (8.3) into (8.1), we have

$$\frac{d^2y}{dx^2} = \frac{M(x)}{EI} \tag{8.4}$$

The equation obtained is a second-order linear differential equation; it is the governing differential equation for the elastic curve.

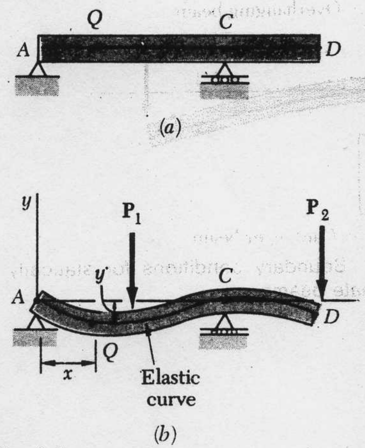


Fig. 8.4

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† It should be noted that, in this chapter and the next, y represents a vertical displacement, while it was used in previous chapters to represent the distance of a given point in a transverse section from the neutral axis of that section.

The product EI is known as the *flexural rigidity* and, if it varies along the beam, as in the case of a beam of varying depth, we must express it as a function of x before proceeding to integrate Eq. (8.4). However, in the case of a prismatic beam, which is the case considered here, the flexural rigidity is constant. We may thus multiply both members of Eq. (8.4) by EI and integrate in x. We write

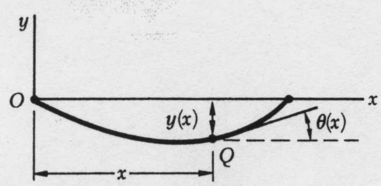


Fig. 8.5

$$EI\frac{dy}{dx} = \int_0^x M(x)\,dx + C_1 \tag{8.5}$$

where C_1 is a constant of integration. Denoting by $\theta(x)$ the angle, measured in radians, that the tangent at Q to the elastic curve forms with the horizontal (Fig. 8.5), and recalling that this angle is very small, we have

$$\frac{dy}{dx} = \tan\theta \simeq \theta(x)$$

Thus, we may write Eq. (8.5) in the alternate form

$$EI\,\theta(x) = \int_0^x M(x)\,dx + C_1 \tag{8.5$'$}$$

Integrating both members of Eq. (8.5) in x, we have

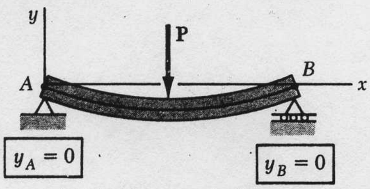
$$EI\,y = \int_0^x \left[\int_0^x M(x)\,dx + C_1\right] dx + C_2$$

$$EI\,y = \int_0^x dx \int_0^x M(x)\,dx + C_1 x + C_2 \tag{8.6}$$

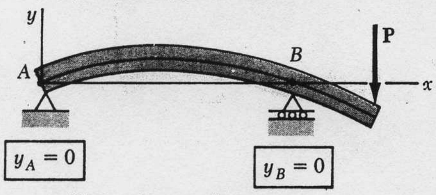
where C_2 is a second constant, and where the first term in the right-hand member represents the function of x obtained by integrating twice in x the bending moment $M(x)$. If it were not for the fact that the constants C_1 and C_2 are as yet undetermined, Eq. (8.6) would define the deflection of the beam at any given point Q, and Eq. (8.5) or (8.5′) would similarly define the slope of the beam at Q.

The constants C_1 and C_2 are determined from the *boundary conditions* or, more precisely, from the conditions imposed on the beam by its supports. Limiting our analysis in this section to *statically determinate beams,* i.e., to beams supported in such a way that the reactions at the supports may be obtained by the methods of statics, we note that only three types of beams need to be considered here (Fig. 8.6): (*a*) the *simply supported beam,* (*b*) the *overhanging beam,* and (*c*) the *cantilever beam.*

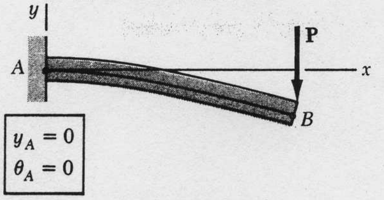
In the first two cases, the supports consist of a pin and bracket at A and of a roller at B, and require that the deflection be zero at each of these points. Letting first $x = x_A$, $y = y_A = 0$ in Eq. (8.6), and then $x = x_B$, $y = y_B = 0$ in the same equation, we obtain two equations which may be solved for C_1 and C_2. In the case of the cantilever beam (Fig. 8.6*c*), we note that both the deflection and the slope at A must be zero. Letting $x = x_A$, $y = y_A = 0$ in Eq. (8.6), and $x = x_A$, $\theta = \theta_A = 0$ in Eq. (8.5′), we obtain again two equations which may be solved for C_1 and C_2.



(*a*) Simply supported beam



(*b*) Overhanging beam



(*c*) Cantilever beam

Fig. 8.6 Boundary conditions for statically determinate beams.