A Game Theoretic Model for Aggregate Lane Change Behavior of Vehicles at Traffic Diverges

Negar Mehr, Ruolin Li, and Roberto Horowitz

Abstract—Lane changes are known to negatively affect traffic delays. However, due to the complexities of this phenomena, accurate and yet simple models of lane change maneuvers are hard to develop. In this work, we present a macroscopic model for predicting the number of vehicles that change lanes at a traffic diverge. We take into account the selfishness of vehicles in selecting their lanes; every vehicle selects lanes such that its own cost is minimized. We discuss how we model such costs experienced by the vehicles. Then, taking into account the selfish behavior of the vehicles, we model lane choice of vehicles at a traffic diverge as a Wardrop equilibrium. We state and prove the properties of our Wardrop equilibrium. We show that there always exists an equilibrium for our model. Moreover, we prove that the equilibrium is unique under mild assumptions. We discuss how our model can be easily calibrated by running a simple optimization problem. Using our calibrated model, we validate our model through simulation studies and demonstrate that our model successfully predicts the aggregate lane change maneuvers at a diverge. We further discuss how our model can be employed to obtain the optimal lane choice behavior of the vehicles, where the social or total cost of the vehicles is minimized.

I. INTRODUCTION

Due to the huge delays and costs that are incurred by traffic congestion, the task of modeling traffic behavior is of paramount importance since such models can be used to analyze traffic networks to get an insight about how the traffic conditions can be improved in urban and freeway networks. However, as traffic networks normally exhibit very complex behaviors, developing models that are both accurate and simple enough for analysis and control purposes is non-trivial. Among the different phenomena that needs to be modeled in traffic networks, lane change behavior of the vehicles has been one of the most challenging behaviors to model. This is due to the fact lane change maneuvers are very complex by nature, and their negative effects on traffic streams are hard to quantify. The existing literature on modeling lane change behaviors is mainly divided into two categories: 1) modeling the microscopic lane change decisions of vehicles, and 2) investigating and quantifying the macro effect of aggregate lane change maneuvers on the traffic conditions.

Related to the first category of the existing research work, the lane change behavior of vehicles was first systematically studied in [7]. In [7], a set of rule and conditions were developed under which a single vehicle was assumed to change lane. The set of derived conditions were assumed to depend on microscopic parameters of the vehicle and road segment such as vehicle velocity and available distance in the neighboring links. Aligned with this, several car following models were proposed by researchers that model the micro behavior of vehicles such as acceleration and lane change maneuvers. Examples of such works can be found in [19], [2], [10]. Due to complexities of such models, researchers studied lane change behavior of vehicles using car following models in simulation studies [8]. Recently, a game theoretic approach was used in [20] and [17] to model the lane change behavior of a single vehicle, where the lane change decision was assumed to be taken by a vehicle for increasing its speed. In [12], similar approach was taken to mimic the behavior of the drivers at traffic merges.

With the recent advances in autonomous vehicles technology, a large body of literature has been devoted on how to design and control autonomous vehicles such that they manifest lane change behaviors that are both safe and similar to the lane change decisions made by humans. In [6], the intention of the drivers when merging to freeway lanes was estimated. In [21], a decision making approach for performing lane changes while driving fully automated in urban environments was presented and evaluated. In [11], the requirements associated with an optimal lane change behavior were described where minimizing fuel consumption and travel time were considered as objectives. In [16], computer vision techniques were utilized to infer lane change intents.

In the second category of the present research work investigating macro effects, there has been a focus on how to quantify the negative effects of lane change maneuvers on upstream traffic congestion. In [15], lane changing vehicles were modeled as particles endowed with mechanical properties. In this work, freeway sections that are away from diverges were considered, and the freeway was modeled as interacting streams which could be linked by these particles. The implications and applications of this model were discussed in [14]. In [4], it was demonstrated via case studies that lane change maneuvers could lead to reductions in freeway capacity. In [9], the impacts of lane change behaviors was modeled via introduction of lane–changing intensity variables and modified fundamental diagrams. In [23], a stochastic lane change model was developed for capturing the system–level lane changing characteristics.

In this paper, we study the aggregate lane change maneuvers taken by the vehicles at traffic diverges. Despite the majority of the existing literature on lane change modeling,
we study it at the macro scale, where we predict the number of vehicles who will change their lane in order to take an appropriate exit that corresponds to their route. In particular, given the number of vehicles who wish to take a certain exit, we develop a novel model which can predict how many vehicles will change their lanes close to the diverge in order to take an exit. We assume that vehicles act selfishly, i.e. every vehicle decides on its route and lane choice such that its own cost is minimized. We describe how the costs incurred on the vehicles can be modeled. Since our focus is on developing a macroscopic fluid–like model for the behavior of the vehicles, we model the equilibrium that results from selfishness of the vehicles as a Wardropian equilibrium. We prove that our model always has an equilibrium, and further, its equilibrium is unique under mild assumptions. We describe how our model can easily be calibrated by solving a mixed-integer linear program, and show through simulation studies that our model shows promising results, it can successfully predict the aggregate lane change behavior of the vehicles at a fork.

Our framework, albeit simple, provides a powerful tool for quantifying the inefficiencies that arise from the selfish lane change behavior of vehicles at traffic diverges. Moreover, our model can be used to determine the optimal aggregate lane change maneuvers of vehicles. It has been shown via multiple simulation studies that lane change can reduce the efficiency of the road, but our framework not only predicts the lane change behavior of the vehicles but it can also be used to quantitatively study and analyze this effect. Our model is particularly beneficial in scenarios when a central authority can route a fraction of vehicles. For instance, in networks with mixed autonomy, autonomous vehicles might be routed by a central planner. In such scenarios, our model can be used for deciding on the optimal lane change behavior of vehicles where the resulting equilibrium has the minimum social cost. To the best of our knowledge, there is no such work in the literature.

The organization of this paper is as follows. In Section II, we describe our modeling framework. In Section III, we state and prove the properties of our model. Simulation studies including model calibration and validation are described in Section IV. In Section V, we describe how our model can be used for choosing the optimal lane change pattern at traffic diverges. We conclude the paper and discuss our future directions in Section VI.

II. THE MODEL

We consider a traffic diverge where a link is bifurcated into two links, which is a common scenario for freeway and arterial forks. We wish to study the route choice behavior of vehicles in such diverges, where certain lanes correspond to a certain route or exit link. Normally, in these scenarios, among the vehicles with the same target exit link, a fraction of vehicles choose the appropriate lanes that correspond to their exit, far upstream of the diverge, while the remaining fraction of the vehicles choose their lane and route very close to the diverge. We wish to obtain a model that given

\[ f_i = x^s_i + x^a_i, \quad \forall i \in I, \]  
\[ \sum_{i \in I} f_i = 1, \]  
\[ f_i \leq 1, \forall i \in I, \]  
\[ x^s_i \geq 0, x^a_i \geq 0, \quad \forall i \in I. \]

**Example 1.** Consider the diverge shown in Figure 1. In this example, there are two freeway lanes I and II which bifurcate to exit links 1 and 2. For this diverge, \( x^s_1 \) is the fraction of vehicles that remain on lane I and take the exit link 1, whereas \( x^a_1 \) is the fraction of vehicles that move along lane II and change their lane from II to I at vicinity of the diverge.

For each \( i \in I \), we assume that all steadfast vehicles constituting \( x^s_i \) experience the same cost. Likewise, all altering vehicles taking an exit link \( i \), experience the same cost. For each destination link \( i \in I \), we let \( J^s_i \) and \( J^a_i \) be the cost incurred on the vehicles forming \( x^s_i \) and \( x^a_i \) respectively. It is important to note that for each \( i \neq j \in I \), \( J^s_i \) or \( J^a_i \) depends not only on \( x^s_i \) and \( x^a_i \) but can also depend on \( x^s_j \) and \( x^a_j \). For each \( i \neq j \in I \), we model the cost of the steadfast
vehicles by
\begin{equation}
J^i_2(x) = C^i_2(x^i_1 + x^i_2) + C^i_1 x^i_1 (x^i_1 + x^i_2),
\end{equation}
where \(C^i_1\) and \(C^i_2\) are positive constants. The constant \(C^i_1\) is the cost of traversing the lanes that correspond to the exit \(i\). Since \((x^i_1 + x^i_2)\) is the total fraction of vehicles that traverse the link that leads to exit \(i\), \((x^i_1 + x^i_2)\) is multiplied by \(C^i_1\) (e.g. \(x^i_1 + x^i_2\) traverses link \(I\) in Figure 1 and the cost is \(C^i_1\)).

Now, consider \(J\) and then change their lane to take the exit \(I\). For this type of vehicles, \(J^i_1(x) = C^i_1(x^i_1 + x^i_2) + C^i_2 x^i_1 (x^i_1 + x^i_2)\). This term is used to mimic the fact that as the vehicles in \(x^i_1\) change their lanes to take the exit \(i\), they use the roads (resources) that join the exit \(i\); thus, they will create delays for the vehicles that are already in their target lanes.

Note that since \(x^i_1\) and \(x^i_2\) both share the target link of \(x^i_1\) up to the vicinity of the divergence, the total fraction of the vehicles present in the target lanes of \(x^i_1\) is \((x^i_1 + x^i_2)\). Hence, \(C^i_1\) is multiplied by \((x^i_1 + x^i_2)\) and \(x^i_1\). This multiplication implies that the higher the number of vehicles that change lanes \((x^i_1)\) is, or, the more occupied the lanes that joins the exist \(i\) is, the larger the incurred cost is.

Now, we describe how we model the costs that the altering vehicles experience. For each \(i \neq j \in I\), we model \(J^i_2\) via
\begin{equation}
J^i_2(x) = C^i_1(x^i_1 + x^i_2) + C^i_2 x^i_1 (x^i_1 + x^i_2),
\end{equation}
where \(\gamma_i\) is a constant assumed to satisfy \(\gamma_i \geq 1\), and \(C^i_1\) and \(C^i_2\) are as previously defined. If \(\gamma_i = 1\), the first and second terms in (6) are the costs that are incurred due to traversing the links that join the exit \(j\) defined by (5). But, if \(\gamma_i > 1\), the additional cost that the altering vehicles must pay due to traversing a longer path for joining their appropriate exit, is modeled. In fact, \(\gamma_i > 1\) can model the cost incurred on altering vehicles due to the additional distance they need to traverse as well as the discomfort cost they will face for changing lanes.

Example 2. Consider the divergence shown in Figure 1. For this example, \(x^i_1\) is the fraction of the vehicles that remain on lane \(I\) and take exit \(I\), whereas \(x^i_2\) is the fraction of the vehicles that use lane \(I\) and leave lane \(I\) close to the divergence to take the exit \(2\). In this case, \(J^i_1(x) = C^i_1(x^i_1 + x^i_2) + C^i_2 x^i_1 (x^i_1 + x^i_2)\). Note that \(C^i_1(x^i_1 + x^i_2)\) is the cost of traversing lane \(I\), where \((x^i_1 + x^i_2)\) is the total fraction of vehicles present on lane \(I\).

We let \(C = (C^i_1, C^i_2, \gamma_i : i \in I)\) be the vector of the cost coefficients in our model. Before proceeding, we need the following definition.

**Definition 1.** A function \(h(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}\) is called monotone if and only if for every \(x, x' \in \mathbb{R}^n\) such that \(x \leq x'\), where inequalities are interpreted elementwise, we have
\[h(x) \leq h(x').\]

Using Equations (5) and (6), the following remark is evident.

**Remark 1.** For each \(i \in I\), the cost functions \(J^i_1\) and \(J^i_2\) are monotone in the sense of Definition 1.

We will later use Remark 1 to guarantee certain properties of our model.

A reasonable and realistic assumption is that vehicles act selfishly, i.e. each vehicle acts such that its own cost is minimized. Each vehicle has two options: to choose its appropriate lane upstream of the divergence or to take its target exit close to the divergence via lane change. Therefore, at equilibrium, if for \(i \in I\), both \(x^i_1\) and \(x^i_2\) are nonzero, we must have \(J^i_1(x) = J^i_2(x)\); otherwise, vehicles will move to the lanes with lower cost. If either \(x^i_1\) or \(x^i_2\) is zero, then, its corresponding cost must be already larger than the one with nonzero flow. These conditions are called Wardrop conditions [22] in the transportation literature. In order to describe the formal definition of Wardrop conditions, let \(G = (F, C)\) be a tuple enclosing \(F\) and \(C\) which are respectively the configuration of demand fractions and the vector of cost coefficients.

In our setting, an equilibrium is defined via the following.

**Definition 2.** For a given \(G = (F, C)\), a flow vector \(x\) is an equilibrium if and only if for every \(i \neq j \in I\), we have:
\begin{align}
x^i_1(J^i_1(x) - J^j_1(x)) &\leq 0, \quad (7a) \\
x^i_2(J^i_2(x) - J^j_2(x)) &\leq 0. \quad (7b)
\end{align}

Note that Equations (7) imply that for an exit link \(i \in I\), if \(x^i_1 \neq 0 \) and \(x^i_2 \neq 0\) at equilibrium, then, we must have that \(J^i_1(x) = J^i_2(x)\). Alternatively, if \(x^i_1 = 0 \) \((x^i_2 = 0)\) at equilibrium, we have \(J^i_1(x) \geq J^i_2(x) \) \((J^i_2(x) \geq J^i_1(x))\). Note that adoption of a Wardrop assumption implies that vehicles can be treated infinitesimally, i.e. the change caused by the unilateral lane change of a single vehicle is negligible. This is in accordance with our goal of modeling the macroscopic behavior of vehicles at diverges.

**III. Equilibrium Properties**

In this section, we state the equilibrium properties of our model including existence and uniqueness.

**A. Equilibrium Existence**

Using the existence Theorem in [3] for the setting of our problem, we can conclude that there always exists at least one equilibrium for a given \(G = (F, C)\) if the following holds.

**Proposition 1.** Given a \(G = (F, C)\) for a diverge, if the cost functions \(J^i_1(x), J^i_2(x), i \in I\) are continuous and monotone
Then, for every equilibrium if and only if
\[ \forall p, p' \in P, \quad y_p = B_p(y_{p'}) \quad \text{(10)} \]
\[ y = (y_i : i \in I) \quad \text{(11)} \]
where \( B_p \) is the best response function of the player \( p \). Note that since \( J(y) \) is a continuous function on any closed interval, thus, the minimum is achieved. Equation (10) implies that if \( y_{p'} \) is fixed, player \( p \) takes its best possible action which is minimizing its own cost \( J_p(y) \). The following proposition establishes the connection between the Wardrop equilibrium of \( G \) and Nash equilibrium of \( \tilde{G} \).

**Proposition 2.** A flow vector \( x = (x_i^t : i \in I) \) is a Wardrop equilibrium for \( G = (F, C) \) if and only if \( y = (y_i^t : i \in I) \) is a pure Nash equilibrium for \( \tilde{G} \) provided that
\[ C_i^t \geq C_i^t, \quad \forall i \in I. \quad \text{(12)} \]

**Proof.** First, note that given the demand fractions \( F = \{f_1, f_2\} \), flow conservation requires that \( x_i^t = f_i - s_i, \forall i \in I \). Thus, with a little abuse of notation, \( J_i^t(x) \) and \( J_i^t(x) \) can be written as \( J_i^t(x_i^t, x_j^t) \) and \( J_i^t(x_i^t, x_j^t) \) for every \( i \neq j \in I \). We show that for every \( i \neq j \in I, \) (12) is a sufficient condition for \( J_i^t(x_i^t, x_j^t) \) to be increasing in \( x_i^t \), and \( J_i^t(x_i^t, x_j^t) \) to be a decreasing function of \( x_j^t \) for any given \( x_i^t \). To see this, note that for a given \( F, f_i \)'s are fixed, and for every \( i \neq j \in I \), we have:
\[ \frac{\partial J_i^t}{\partial x_i^t} = -2C_i^t x_i^t + C_i^t + C_i^t(x_i^t) - C_i^t(f_j - x_j^t). \quad \text{(13)} \]
Equation (13) is linear in \( x_i^t \). Moreover, for each \( i \in I \), \( x_i^t \) is allowed to only take values in the interval \([0, f_i] \). Therefore, in order to obtain sufficient conditions for the positivity of (13), it is sufficient to guarantee that for every \( i \neq j \in I \), \( \frac{\partial J_i^t}{\partial x_i^t} \) is positive at all possible extreme points \( (x_i^t, x_j^t) \) which are \( \{(0, 0), (f_1, 0), (0, f_2), (f_1, f_2)\} \). Using Equation (3), it is easy to verify that the smallest possible value of (13) might be attained in \( (f_1, 0) \) where \( f_1 = 1 \). At the point \((1, 0)\), we have \( \frac{\partial J_i^t}{\partial x_i^t}(1, 0) = C_i^t - C_i^t \). Therefore, (12) is a sufficient condition for \( J_i^t \) to be increasing in \( x_i^t \). Similarly, we can
compute $\frac{\partial J_i}{\partial x_i}$. For every $i \neq j \in I$

$$
\frac{\partial J_i}{\partial x_i} = -C_i'\gamma_j - C_j'(f_j - x_j^p).
$$

(14)

Since $(f_j - x_j^p)$ is always greater than or equal to zero, clearly, for every $i \neq j \in I$, $J_i'(x_j^p, x_j^p)$ is always decreasing in $x_i^p$ for any given $x_j^p$.

Now, consider the best response function $B_p(y_p')$ in (10). In order to minimize $J_p(y_p, y_p')$ over $y_p$ as the best response for a given $y_p'$, with $y_p = x_i^p$ and $y_p' = x_j^p$, since $\frac{\partial J_i}{\partial x_i}$ is increasing, and $\frac{\partial J_i}{\partial x_i}$ is decreasing in $x_j^p$ under (12), the following scenarios might occur for a given $x_j^p$, where $i \neq j \in I$ (see Figure 2).

- **Case A**: $J_i'(x_i^p, x_j^p)$ and $J_i'(x_i^p, x_j^p)$ have an intersection on the interval $[0, f_i]$. In this case, there exists a point $x_i^p(x_j^p) \in [0, f_i]$ such that $J_i'(x_i^p, x_j^p) = J_i'(x_i^p, x_j^p)$. Using (9), it is easy to see that in this case, $y_p = x_i^p$ is the best response for a given $y_p = x_i^p$ in the game $G$. If this is the case, Equations (7) are also satisfied by $x_i^p$ for a given $x_j^p$. It is easy to see that the reverse is also true. Indeed, if $x_i^p$ satisfies (7) for the given $x_j^p$, then $x_i^p$ must be the intersection of $J_i'(x_i^p, x_j^p)$ and $J_i'(x_i^p, x_j^p)$ on the interval $[0, f_i]$. Therefore, $y_p = x_i^p$ is the best response of $y_p = x_i^p$.

- **Case B**: $J_i'(x_i^p, x_j^p)$ and $J_i'(x_i^p, x_j^p)$ do not intersect on the interval $[0, f_i]$, and $J_i'(0, x_i^p) \geq J_i'(0, x_j^p)$ for a given $x_j^p$. In this case, if $y_p' = x_j^p$, then $y_p = B_p(y_p') = 0$ (See Figure 2, case B). It is easy to see that, $x_i^p = 0$ satisfies (7) for a given $x_j^p$, but $y_p = x_i^p$ while $J_i'(0, x_i^p) \geq J_i'(0, x_j^p)$. The reverse is also true, if $x_i^p = 0$ satisfies (7) for a given $x_j^p$, $y_p = x_i^p = 0$ is the best response of $y_p' = x_j^p$.

- **Case C**: $J_i'(x_i^p, x_j^p)$ and $J_i'(x_i^p, x_j^p)$ do not intersect on the interval $[0, f_i]$, and $J_i'(0, x_i^p) \leq J_i'(0, x_j^p)$. In this case, $y_p' = x_j^p$, $y_p = B_p(y_p') = 1$. Similar to the case B, one can conclude that if $y_p' = x_j^p$, $y_p = x_i^p = 1$ is equal to $B_p(y_p')$ if and only if $x_i^p = 1$ satisfies (7) for a given $x_j^p$. So far, we have shown that for every $p \neq p' \in P$, given $y_p'$, $y_p = B_p(y_p')$ if and only if $x = (y_p, f_i - y_p : i \in I)$ satisfies (7). Therefore, $y = (x_i^p, i \in I)$ is a Nash equilibrium of $G$ if and only if $x = (x_i^p, f_i - x_i^p)_{i \in I}$ is a Wardrop equilibrium of $G$.

Remark 3. Notice that using the three cases described in the proof of Proposition 2, for a given $y_p' = x_j^p$, $B_p(y_p')$ can be found by first intersecting $J_i'(x_i^p, x_j^p)$ and $J_i'(x_i^p, x_j^p)$ and then projecting their intersection $x_i^p(x_j^p)$ onto the interval $[0, f_i]$. We will use this fact in the remainder to prove equilibrium uniqueness.

Having Proposition 2 in mind, we are ready to state and prove the following.

**Theorem 1.** For a given $G = (F, C)$, the Wardrop equilibrium flow vector $x$ is unique if

$$
C_i^p \geq C_i^c, \forall i \in I,
$$

(15)

$$
(\gamma_i - 1)C_i^j \geq C_i^c, \forall i \in I.
$$

(16)

**Proof.** Construct the auxiliary game $\tilde{G} = (P, A, (\tilde{J}_p, p \in P))$ described above from $G$. Using Proposition 2, we know that if (15) holds, $x$ is a Wardrop equilibrium for $G$ if and only if $(y_p : p \in P) = (x_i^p : i \in I)$ is a Nash equilibrium for $\tilde{G}$. We now prove that under (16), $\tilde{G}$ has a unique equilibrium; thus, $G$ must also have a unique equilibrium if (15) and (16) hold. To see this, note that using (10), $y = (x_i^p : i \in I)$ is a Nash equilibrium for $\tilde{G}$ if and only if for every $p \neq p'$, $y_p = B_p(y_p')$, and $y_p' = B_p(y_p')$. These conditions can be rewritten as

$$
y_p = B_p(B_p(y_p)),
$$

(17a)

$$
y_p' = B_p(B_p(y_p')).
$$

(17b)

Equations (17) indicate that $y$ is an equilibrium if and only if for every $p \neq p' \in P$, $y_p$ is a fixed point for $B_p(B_p(\cdot))$. Thereby, $(y_p, y_p')$ is an equilibrium for $\tilde{G}$ if and only if $B_p(B_p(\cdot)))$ intersects the line going through the origin with slope 1, at $y_p$. In the remainder, we prove that under (16), the slope of $B_p(B_p(\cdot))$ is always positive and smaller than 1 for every $p \neq p' \in P$. Therefore, $B_p(B_p(\cdot))$ can intersect the identity line at most once. Thus, we can then conclude that $G$ and therefore $\tilde{G}$ will always have a unique equilibrium if (15) and (16) hold. To prove this, it suffices to prove that $0 \leq \frac{\partial J_i^p}{\partial x_i^p} \leq 1$, for every $p \neq p' \in P$. To see this, let $x_j^p$ be such that $J_i'(x_i^p, x_j^p)$ and $J_i'(x_i^p, x_j^p)$ intersect each other at $x_i^p(x_j^p) \in [0, f_i]$. Using (5), and the fact that $x_i^p = f_i - x_i^p$, $\forall i \in I$, we have that

$$
\frac{\partial J_i^p}{\partial x_i^p} = -C_i^p - C_i^c(f_j - x_i^p).
$$

Since $(f_j - x_i^p) \geq 0$, we can conclude that $\frac{\partial J_i^p}{\partial x_i^p} \leq 0$. Similarly, we can compute

$$
\frac{\partial J_i^p}{\partial x_j^p} = C_i^p + C_i^c(f_j - x_i^p) - C_i^c(x_j^p + f_i - x_i^p).
$$

Since $(f_j - x_i^p) \geq 0$, it is easy to see that if (15) holds, $\frac{\partial J_i^p}{\partial x_j^p}$ is always positive. This implies that as $x_j^p$ increases, $J_i^p(x_i^p, x_j^p)$ decreases while $J_i^p(x_i^p, x_j^p)$ increases. Therefore, as $x_j^p$ increases, $\frac{\partial J_i^p}{\partial x_j^p}$ can only increase. However, Remark 3 implies that if $x_i^p(x_j^p)$ lies outside the interval $[0, f_i]$, it is projected on this interval. Thus, the interval $[0, f_i]$ can be divide into three intervals $[0, f_i] = [m_i, n_i] \cup [m_i, n_i] \cup [m_i, f_i]$, such that $x_i^p(x_j^p)$ is always 0 on $[m_i, n_i]$, and always 1 on $[n_i, f_i]$. Note that either of the intervals $[0, m_i]$, $[m_i, n_i]$ and $[n_i, f_i]$ can possibly be empty. Hence, in order to show that the slope of the best response function $B_p(\cdot)$ is always smaller than 1 it suffices to show it for the interval $[m_i, n_i]$ where $J_i'(x_i^p, x_j^p)$ and $J_i'(x_i^p, x_j^p)$ do intersect.

On the interval $[m_i, n_i]$, for a given $x_j^p$, $x_i^p(x_j^p)$ must satisfy:

$$
J_i'(x_i^p, x_j^p) - J_i'(x_i^p, x_j^p) = 0.
$$
Therefore, using implicit differentiation, \( \frac{d\bar{x}_a(x_j^a)}{dx_j} \) can be computed via

\[
\frac{\partial}{\partial x_i} \left( J_i^a(\vec{x}_i^a, x_j^a) - J_i^a(\vec{x}_i^a, x_j^a) \right) \frac{d\bar{x}_a(x_j^a)}{dx_j} + \frac{\partial}{\partial x_j^a} \left( J_i^a(\vec{x}_i^a, x_j^a) - J_i^a(\vec{x}_i^a, x_j^a) \right) = 0. \tag{18}
\]

Using (5) and (6) and flow conservation \( x_i^a = f_i - x_i^a \) for all \( i \in I \), we have

\[
\frac{\partial}{\partial x_j^a} \left( J_i^a(\vec{x}_i^a, x_j^a) - J_i^a(\vec{x}_i^a, x_j^a) \right) = -C_i^a - C_j^a(f_i - x_i^a) - C_i^a + C_j^a(x_j^a + f_i - x_i^a) - C_j^a(f_j - x_j^a). \tag{19}
\]

Since (19) is linear in \( x_i^a \) and \( x_j^a \), its maximum and minimum are attained in its extreme points. It is easy to check that the maximum possible value for (19) is \( -C_i^a - C_j^a \). If (15) holds, \( -C_i^a - C_j^a + C_j^a \leq 0 \). Therefore \( \frac{\partial}{\partial x_j} \left( J_i^a(\vec{x}_i^a, x_j^a) - J_i^a(\vec{x}_i^a, x_j^a) \right) \leq 0 \) under (15). Using the same trick of computing the function at its extreme points, it can be easily seen that under (15), it always the case that for every \( i \in I \),

\[
\frac{\partial}{\partial x_j^a} \left( J_i^a(\vec{x}_i^a, x_j^a) - J_i^a(\vec{x}_i^a, x_j^a) \right) \geq 0.
\]

Hence, using (18), under (15),

\[
\frac{d\bar{x}_a(x_j^a)}{dx_j} \geq 0, \quad \forall i \neq j \in I.
\]

Now that we have shown that the slope of the best response function is always positive, it only remains to prove that \( \frac{d\bar{x}_a(x_j^a)}{dx_j} \leq 1 \). To prove this, it suffices to show that

\[
\frac{\partial}{\partial x_i^a} \left( J_i^a(\vec{x}_i^a, x_j^a) - J_i^a(\vec{x}_i^a, x_j^a) \right) \geq -\left( \frac{\partial}{\partial x_j^a} \left( J_i^a(\vec{x}_i^a, x_j^a) - J_i^a(\vec{x}_i^a, x_j^a) \right) \right). \tag{20}
\]

Substituting (5), (6), and (19) in (20) and computing the linear function at its extreme points, we observe that (16) is a sufficient condition for (20).

### IV. Simulation Results

Up to now, we have described our modeling framework and the properties of our model. In this section, we describe how our simulation results indicate that our model can successfully predict the observed behaviors. A key element of our model which affects its functionality is the vector \( C \). In order to study the performance of our model, first, the model needs to be calibrated, i.e. the \( C \) that best fits a diverge must be found.

#### A. Model Calibration

Consider a diverge with two exit links \( I = \{1, 2\} \). Fix the total flow of vehicles \( d = d_1 + d_2 \) that enter the diverge. For the fixed \( d \), consider different configurations of demand fractions, \( F^k = \{f_1^k, f_2^k\}, 1 \leq k \leq K \), where \( K \) is the total number of possible demand fraction configurations available from the data or simulation. For each value of \( f_1^k \) and \( f_2^k \), record \( (x_i^a)^k \) and \( (x_j^a)^k \) which are the fraction of steadfast and altering vehicles for each destination \( i \in I \) when the \( k \)'th demand pattern is used. We let \( x^a \) represent the vector \( x \) for the \( k \)'th demand configuration. Using our model, the vector of cost coefficients \( C \) must be found such that (7) is satisfied by \( (x_i^a)^k \) and \( (x_j^a)^k \) for every \( k \leq K \). But since (7) contains nonlinear inequalities, finding such a \( C \) is nontrivial. We propose the following for calibrating \( C \).

For every \( k \leq K \) and \( i \in I \), define the integer variables \( (z_i^a)^k \in \{0, 1\} \), and \( (z_j^a)^k \in \{0, 1\} \) such that:

\[
(x_i^a)^k(J_i^a(x^k) - J_i^a(x^0)) \leq 0 \iff (z_i^a)^k = 0 \tag{21a}
\]
\[
(x_j^a)^k(J_j^a(x^k) - J_j^a(x^0)) > 0 \iff (z_j^a)^k = 1 \tag{21b}
\]
\[
(x_i^a)^k(J_i^a(x^k) - J_i^a(x^0)) \leq 0 \iff (z_i^a)^k = 0 \tag{21c}
\]
\[
(x_j^a)^k(J_j^a(x^k) - J_j^a(x^0)) > 0 \iff (z_j^a)^k = 1 \tag{21d}
\]

Then, we propose to solve the following optimization problem for calibrating \( C \).

\[
\text{minimize} \quad \sum_{k \in K} \sum_{i \in I} ((z_i^a)^k + (z_j^a)^k) \quad \text{subject to Equations (21)} \tag{22}
\]
\[
C_j \geq 1,
\]

where \( C_j \) is the \( j \)'th element of \( C \). We use the constraint \( C_j \geq 1 \) to avoid setting all the elements of \( C \) to be zero. It is important to note that since in (5) and (6), every term is multiplied by one and only one element of \( C \), and also, multiplying all the cost functions by the same constant does not change the Wardrop conditions, scaling \( C \) by a single number will not affect the model. Therefore, this constraint does not affect the model. Note that for every inequality constraint that is violated in (22), the cost is increased by 1. Thus, (22) penalizes for not satisfying (7) which are the equilibrium conditions. But, how can the problem optimization (22) be solved where the constraints are of the form (21)? To answer this, we use the procedure introduced in [18]. Let \( M \) be a large positive number, and \( \epsilon \) be a small positive number close to zero. For every \( k \), the following is equivalent to (21).

\[
(x_i^a)^k(J_i^a(x^k) - J_i^a(x^0)) \leq M(z_i^a)^k - \epsilon, \tag{23a}
\]
\[
-(x_i^a)^k(J_i^a(x^k) - J_i^a(x^0)) \leq M(1 - z_i^a)^k - \epsilon, \tag{23b}
\]
\[
(x_j^a)^k(J_j^a(x^k) - J_j^a(x^0)) \leq M(z_j^a)^k - \epsilon, \tag{23c}
\]
\[
-(x_j^a)^k(J_j^a(x^k) - J_j^a(x^0)) \leq M(1 - z_j^a)^k - \epsilon. \tag{23d}
\]
Therefore, our model can be calibrated by solving the following optimization problem:

\[
\begin{align*}
\text{minimize} & \quad \sum_k \sum_{i \in I} (z_i^* + (z_i^*)^k) \\
\text{subject to} & \quad \text{Equations (23), (24)} \\
& \quad C_j \geq 1.
\end{align*}
\]

Note that (24) is now a mixed-integer linear program that can be easily solved using optimization packages. Since (24) is solved offline, and, further, the number of required integer variables is small, the computational complexities for solving (24) are not overtaxing in our model.

### B. Model Validation

Consider the diverge shown in Figure 1. We used the microscopic traffic simulator SUMO [13] to simulate the traffic behavior at the diverge of Figure 1 for different demand configurations. A total flow of \(d_1 + d_2 = 3000 \text{veh hour}^{-1}\) enters the diverge. The capacity of every lane is 930 \text{veh hour}^{-1}. At every simulation, a fraction of vehicles \(f_1\) is assumed to take the exit link 1 while the remaining fraction of vehicles \(f_2 = 1 - f_1\) is assumed to take the exit link 2. For different values of \(f_1, x_1^*, x_2^*, f_2, \) and \(x_2^*\) are measured. Then, this data set is used to calibrate the model, i.e., finding the \(C\) that best fits the data. By solving the optimization problem (24), we found the values of \(C\). Since our road geometry is symmetric, we introduced the additional constraints that \(C_1^e = C_2^e\), \(C_1^c = C_2^c\), and \(\gamma_1 = \gamma_2\) in (24), and obtained the following values for \(C\):

\[C_1^e = C_2^e = 1, \quad C_1^c = C_2^c = 1, \quad \gamma_1 = \gamma_2 = 2.7.\]

Note that the obtained values of \(C\) satisfy (15) and (16); thus, in every scenario, Theorem 1 implies that there exits only one equilibrium. The objective function of (24) was 4 when fitting \(C\), meaning that only 4 inequalities were unsatisfied among our data set.

With the calibrated \(C\), we used our model to predict \(x_1^*, x_1^*, x_2^*, \) and \(x_2^*\) for the scenarios where the total flow entering the diverge is different from the one we used in calibration. Figure 3 demonstrates such a study for the case where the total flow of the diverge is 25000 \text{veh hour}^{-1}. Figure 3 shows both our simulation results and our model prediction for different configuration of the demands of exit links. As Figure 3 shows, our model can successfully predict the fraction of altering vehicles for each destination. Note that when the demand for exit 1 is low \(f_1 \leq 0.5\), none of the vehicles who aim to take the exit 1 would take the more crowded lane II; therefore, \(x_1^* \approx 0\). But, with the increase of \(f_1\), vehicles will take lane II since it will reduce their cost. Our simulation results indicate that our model is capable of predicting with great accuracy the behavior of the vehicles. We obtained similar results when the total flow that enter the diverge was varied.

### V. Socially Optimal Lane Change Behavior

Having shown that our model can lead to promising results, we can deploy it for further analysis. Intuitively, one might argue that if vehicles were less selfish, and would have chosen their destination lane far upstream of the diverge, it would have reduced the total cost of the vehicles. But how can we quantify this? Our model provides a powerful framework for analytically studying this conjecture. Assume that there is a central authority which can dictate vehicle lanes such that the total cost of the vehicles is minimum, or equivalently, that the social optimum is achieved. How would such authority pick lanes for the vehicles? To answer this question, using our model, we can solve the following optimization problem

\[
\begin{align*}
\text{minimize} & \quad \sum_{i \in I} (x_i^s J_i^s + x_i^o J_i^o) \\
\text{subject to} & \quad x_i^s + x_i^o = f_i, \quad \forall i \in I, \\
& \quad x_i^s \geq 0, \quad x_i^o \geq 0, \quad \forall i \in I.
\end{align*}
\]

Optimization (25) can be solved to find the optimal lane change behavior. Note that in (25), the decision variables are \(x_i^s, x_i^o, \forall i \in I\); thus, (25) is a quadratic program which can be easily solved. This simplicity is in contrast to the existing models, where strategies for finding better lane choices are heuristically proposed through simulation.

Using the \(C\) that was obtained from our model calibration, we solved (25) for the case when the total flow entering the diverge is 3000 \text{veh hour}^{-1}. Figure 4 demonstrates the optimal lane choice of the vehicles. As Figure 4 shows, since people choose their lanes selfishly, at equilibrium, the number of
altering vehicles is larger than the optimal one. Moreover, as Figure 4 suggests, a key observation is that the optimal lane choice is not preventing all vehicles from changing lanes. Therefore, the optimal lane choice is in between the equilibrium and zero lane change. Our model can be used for quantitatively obtaining this trade-off, which has not been captured in previous studies.

VI. CONCLUSION AND FUTURE WORK

We provided a game theoretic framework for macroscopically modeling the aggregate lane change maneuver of vehicles at traffic diverges. We modeled the fraction of vehicles who change lanes to take an exit, where vehicles were assumed to be selfish. We modeled the resulting equilibrium as a Wardrop equilibrium and proved the existence and uniqueness of this equilibrium. We described how our model can be easily calibrated and demonstrated via simulation studies that our model yielded promising results. For future steps, we are excited about the applications this model can have. For instance, when an authority might have control over a fraction of vehicles, we can study how to enforce lanes such that resulting equilibrium has lower social cost.

ACKNOWLEDGMENTS

This work is supported by the National Science Foundation under Grants CPS 1446145 and CPS 1545116.

REFERENCES


