

Passive Velocity Field Control (PVFC): Part I—Geometry and Robustness

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Abstract—Passive velocity field control (PVFC) is a control methodology for fully actuated mechanical systems, in which the motion task is specified behaviorally in terms of a velocity field (as opposed to a timed trajectory), and the closed-loop system is passive with respect to a supply rate given by the environment power input. It is intended for safety critical and coordination intensive applications where the mechanical system is required to interact with the physical environment. The control law is derived geometrically and the geometric and robustness properties of the closed-loop system are analyzed. It is shown that the closed-loop unforced trajectories are geodesics of a *closed-loop connection* which is compatible with an inertia metric, and that the velocity of the system converges exponentially to a scaled multiple of the desired velocity field. The robustness property of the system exhibits some strong directional preference. In particular, disturbances that push in the direction of the desired momentum do not adversely affect performance. Moreover, robustness property also improves with more energy in the system. In Part II of the paper, the application of PVFC to contour following as well as experimental results are presented.

Index Terms—Affine connections, geodesics, mechanical systems, passivity, safety, velocity field.

I. INTRODUCTION

A CONTROL methodology known as passive velocity field control (PVFC) was recently proposed in [14] for fully actuated mechanical systems. This control scheme has two distinct features: 1) the desired behavior of the mechanical system is specified in terms of velocity fields defined on the configuration manifold of the system. This is in contrast to the traditional method of specifying a task as a desired timed trajectory tracking problem; 2) the mechanical system under closed-loop control appears to be an energetically passive system to its physical environments (i.e., dissipative with respect to the supply rate given by the mechanical power input by the environment). The motivations for developing PVFC are to tackle robotic applications that require a) intimate interaction between the machine and uncertain physical environments (such as humans and other objects liable to be damaged); and b) the coordination between the various degrees of freedom of the machine for the task to be accomplished. By requiring energetic passivity, coupling stability [6], and safety of the machine as it interacts with humans

and physical objects will be enhanced. Indeed, PVFC was initially developed to meet the safety needs in control systems for smart exercise machines [12], [13]. The motivation for specifying tasks via velocity fields is to explicitly emphasize the coordination aspect of some tasks over time keeping. For example in contour or path following, which is an important task for machining operations, the precise coordination between the various degrees of freedom may be considered more critical than keeping in step with any timed trajectory.

When PVFC was first presented in [11], [14], it was formulated in local coordinate notations for ease of access by the robotics community, at the expense that interesting geometric insights cannot be easily brought out. Moreover, in [11], [14], only the basic convergence results were presented. The present paper focuses on the geometric and robustness properties of PVFC. The problem definition, controller derivation and analysis are all done using intrinsic geometric objects. This avoids the need for a prejudicial choice of coordinates and the concern that the controller properties may be functions of this choice. In this geometric framework, geometric properties of the closed-loop system will be illustrated. Robustness results not contained in [11], [14] will also be presented. Before applying PVFC to an application, an appropriate velocity field must first be defined. This step is highly task dependent and may not be trivial. In the companion paper [15], we develop a procedure for applying passive velocity field control to a class of contour following problems. In addition, experimental results which are instructive for the understanding of the properties of PVFC will also be presented.

As is well known, the dynamics of an open-loop mechanical system can be conveniently described by the Levi-Civita connection associated with its inertia metric [4], [9], so that the unforced trajectories are geodesics of the connection. The Levi-Civita connection is the unique affine connection defined on the configuration space of the mechanical system which is both compatible with the inertia metric and is torsion free. Compatibility gives rise to conservation of energy for a mechanical system. When the mechanical system is under passive velocity field control, it turns out that one can define a new affine connection, which we refer to as the closed-loop connection, such that its geodesics are in fact the unforced trajectories (i.e., the trajectories of the system in the absence of environment forces) of the closed-loop system. The closed-loop connection, albeit not Levi-Civita, is also compatible with the inertia metric of an augmented system, giving rise once again to the passivity property of the closed-loop system. Moreover, when one of the feedback gains is zero, the closed-loop connection admits the desired velocity field (which encodes the task) as a parallel vector field.

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The robustness properties of the closed-loop system under PVFC has some interesting characteristics. Specifically, the effect of environment forces on the performance of the closed-loop system to follow a scaled copy of the desired velocity field depends strongly on the direction of the environment forces and on whether they dissipate or inject energy. Environment forces in the direction of the desired momentum or injects energy into the system do not adversely affect performance. Somewhat unexpectedly, the robustness to environment disturbances actually improves at higher speed of operation!

The rest of this paper is organized as follows. The PVFC objective is stated in Section II. In Section III, we review some mathematical preliminaries. The passive velocity field control algorithm is derived in Section IV. The closed-loop control system is analyzed in terms of its geometric structure in Section V. Section VI reports on the robustness properties of the control scheme. Section VII contains some concluding remarks.

Notations: Throughout this paper, unbold letters or symbols denote elements in a manifold, respective bold letters and symbols denote their coordinate representations. Alternate “up” and “down” indices are to be summed implicitly unless otherwise noted. Thus $\dot{\mathbf{q}} = [\dot{q}^1, \dot{q}^2, \dots, \dot{q}^n]^T$ is the coordinate representation of $\dot{q} \in T_q\mathcal{G}$ meaning that $\dot{q} = \dot{q}^i(\partial/\partial q^i)$ or explicitly, $\sum_{i=1}^n \dot{q}^i(\partial/\partial q^i)$ where $\partial/\partial q^i$ are the basis vectors for $T_q\mathcal{G}$. Inner products are denoted by $\langle\langle \cdot, \cdot \rangle\rangle$ and $\langle F, v \rangle$ denotes the action of the co-vector (or a one form) $F \in T_q^*\mathcal{G}$ ($F: \mathcal{G} \rightarrow T^*\mathcal{G}$) on a vector (or a vector field) $v \in T_q\mathcal{G}$ ($v: \mathcal{G} \rightarrow T\mathcal{G}$).

II. PVFC PROBLEM

We consider a fully actuated Euler–Lagrange mechanical system with a n dimensional configuration manifold \mathcal{G} and a Lagrangian given by the system’s kinetic energy. $q(t) \in \mathcal{G}$ and $\dot{q}(t) \in T_{q(t)}\mathcal{G}$ will be used to denote the configuration and velocity of the system at time t . Let \mathcal{G} be equipped with a Riemannian metric M which assigns a family of inner products $\langle\langle \cdot, \cdot \rangle\rangle: T\mathcal{G} \times T\mathcal{G} \rightarrow \mathbb{R}$. We can also think of M as a (1,1) tensor, $M: T\mathcal{G} \rightarrow T^*\mathcal{G}$ such that $\forall v, w \in T_q\mathcal{G}$, and $\forall q \in \mathcal{G}$,

$$\langle\langle M(v), w \rangle\rangle = \langle\langle v, w \rangle\rangle.$$

M is represented by the symmetric inertia matrix, $\mathbf{M}: \mathcal{G} \rightarrow \mathbb{R}^{n \times n}$ with elements:

$$M_{ij} = \left\langle\left\langle \frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j} \right\rangle\right\rangle.$$

We assume that M defines the kinetic energy, $\kappa: T\mathcal{G} \rightarrow \mathbb{R}$ and the Lagrangian $L: T\mathcal{G} \rightarrow \mathbb{R}$ of the mechanical system via

$$\kappa(\dot{q}) = L(\dot{q}) = \frac{1}{2} \langle\langle \dot{q}, \dot{q} \rangle\rangle = \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{M}(\mathbf{q}) \dot{\mathbf{q}}. \quad (1)$$

It is assumed that the system is subject to both controlled actuation $T(t) \in T_{q(t)}^*\mathcal{G}$ (motor torques) and uncontrolled environment forces $F_e(t) \in T_{q(t)}^*\mathcal{G}$. Moreover, we assume that the mechanical control system is *fully actuated*, meaning that $T(t) \in T_{q(t)}^*\mathcal{G}$ can be arbitrarily assigned. We also assume that

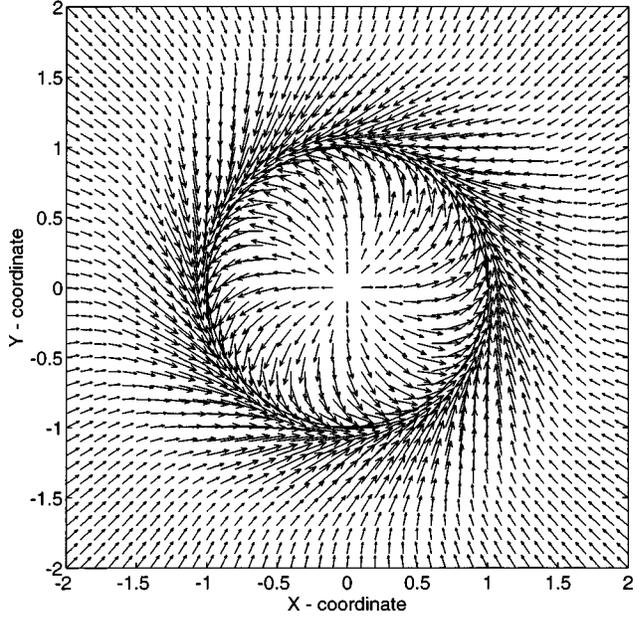


Fig. 1. A velocity field for a two degree of freedom mechanical system with a rectangular configuration manifold tracing a circle.

included in the environment force F_e , are the potential and dissipative forces in addition to friction and environment contact forces.

The dynamics of such a system will be described later in a coordinate free manner using the geometric concept of affine connection. For the moment, the Euler-Lagrange equation for such a system is given by

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = T_i + F_{e,i}, \quad i = 1, \dots, n. \quad (2)$$

A. Using Velocity Fields to Specify Tasks

In PVFC, the control task is specified by a desired velocity field $V: \mathcal{G} \rightarrow T\mathcal{G}$ which defines a desired velocity vector at each configuration. As an example, Fig. 1 illustrates a velocity field for the task of tracing a circular contour on a rectangular configuration space with the arrows representing the desired velocities. Notice that the flow of the velocity field which is determined by tracing the arrows converges to the circle. In general, a contour following task is specified by a velocity field that has the following properties:

- 1) its value at each point of the contour is tangent to the contour;
- 2) the flow of the field has a limit set that is contained in the contour.

Define the α velocity field tracking error by

$$e_\alpha(t) = \dot{q}(t) - \alpha V(q(t))$$

where $q(t)$ and $\dot{q}(t)$ are the configuration and velocity of the system at time t and α is some positive scaling factor. If $e_\alpha(t)$ can be made to vanish for some $\alpha > 0$, the contour will be asymptotically followed. Since the mechanical system is not required to be at a particular position at a particular time, an appropriately designed velocity field directs the system from its

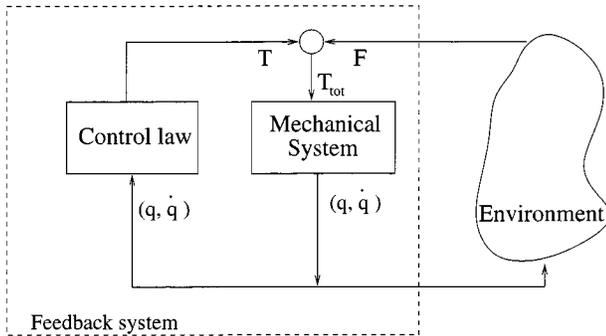


Fig. 2. A mechanical control system interacts with the environment and control. In PVFC, the input–output system within the dotted line must be passive.

current configuration, to approach the contour in a well behaved manner. The constant α determines the traversal speed. The design of the desired field $V : \mathcal{G} \rightarrow T\mathcal{G}$ is task dependent and may not be trivial. Even for contour following tasks, such a velocity field may not exist. How PVFC can be applied to general contour following tasks will be the topic of the companion paper [15].

B. Passivity

Recall that a dynamic system with input $u \in \mathcal{U}$ and output $y \in \mathcal{Y}$ is *passive* with respect to the supply rate $s : \mathcal{U} \times \mathcal{Y} \rightarrow \mathfrak{R}$ if, for any $u : \mathfrak{R}^+ \rightarrow \mathcal{U}$,

$$\int_0^t s(u(\tau), y(\tau)) d\tau \geq -c^2$$

where $c \in \mathfrak{R}$ depends on the initial conditions. In this paper, we are interested in supply rates which are directly related to mechanical power. Specifically, when the control input $T(t)$ is being determined by a control law (Fig. 2), the supply rate that represents the mechanical power input by the environment forces is:

$$s(F_e, \dot{q}) := \langle F_e, \dot{q} \rangle.$$

C. Control Objective

We are now ready to define the PVFC Problem.

Let $V : \mathcal{G} \rightarrow T\mathcal{G}$ be a desired velocity field on \mathcal{G} which has been designed to capture the specified task for the system. Find a control law for T such that

- 1) the input–output system with F_e as input and \dot{q} as output is passive with respect to the supply rate $s(F_e, \dot{q}) = \langle F_e, \dot{q} \rangle$;
- 2) In the absence of environment force ($F_e = 0$), for any initial condition $\dot{q}(0)$, there exists a constant $\alpha > 0$ such that the α -velocity error vanishes

$$\lim_{t \rightarrow \infty} e_\alpha(t) = \lim_{t \rightarrow \infty} [\dot{q}(t) - \alpha V(q(t))] = 0.$$

The scaling factor α is not specified *a priori*, but will be determined by the amount of energy in the system. Notice that time is not explicitly used in the problem definition and that $q(t)$ is not required to track exactly any specified timed trajectory.

III. MATHEMATICAL PRELIMINARIES

A differential geometric framework will be utilized in this paper. The dynamics of a mechanical control system will be described using the concept of *affine connection* from Riemannian geometry. We review some of the necessary elements in this section. For more detailed background in differential geometry and geometric mechanics, the readers are referred to texts such as [2], [3], [7], [17], and [1] [16], respectively.

A. Affine Connection

An *affine connection*, ∇ on \mathcal{G} smoothly assigns to each pair of vector fields $X, Y : \mathcal{G} \rightarrow T\mathcal{G}$, another vector field, $\nabla_X Y$ (called the *covariant derivative of Y w.r.t. X*) on \mathcal{G} satisfying the following scaling and derivation properties: for any smooth function $f : \mathcal{G} \rightarrow \mathfrak{R}$,

$$\nabla_{fX} Y = f \nabla_X Y \quad (3)$$

$$\nabla_X fY = f \nabla_X Y + (\mathcal{L}_X f)Y \quad (4)$$

where $\mathcal{L}_X f$ denotes the Lie derivative of f with respect to X .

Let $(q^1, \dots, q^n) : \mathcal{G} \rightarrow \mathfrak{R}^n$ be a set of local coordinate functions. An affine connection ∇ defines, and is locally defined by n^3 real valued functions $\Gamma_{ij}^k : \mathcal{G} \rightarrow \mathfrak{R}$, $i, j, k = 1, \dots, n$ (the coefficients of ∇) given by:

$$\nabla_{\partial/\partial q^i} \frac{\partial}{\partial q^j} = \Gamma_{ij}^k \frac{\partial}{\partial q^k}. \quad (5)$$

Roughly speaking, $(\nabla_X Y)(q_0)$ defines a kind of directional derivative of $Y : \mathcal{G} \rightarrow T\mathcal{G}$ at q_0 along an integral curve of the vector field X . ∇ supplies the extra relationship between tangent fibers $T_q \mathcal{G}$ at different q 's, so that they can be compared and differentiated. The directional derivative interpretation suggests that $\nabla_X Y(q_0)$ should depend only on $X(q_0)$ and the values of Y on a short integral curve segment of $X(q)$, i.e., $\{q(\tau) : \tau \in (t - \epsilon, t + \epsilon)\}$ with $\epsilon > 0$, $\dot{q}(\tau) = X(q(\tau))$ and $q(t) = q_0$. This can be verified by computing the coordinate representation of $\nabla_X Y$. Let $X(q) = X^i(q) \partial/\partial q^i$ and $Y(q) = Y^j(q) \partial/\partial q^j$. Then using the defining properties of a connection (3)–(5)

$$(\nabla_X Y)(q_0) = \{(\mathcal{L}_X Y^k)(q_0) + \Gamma_{ij}^k(q_0) X^i(q_0) Y^j(q_0)\} \frac{\partial}{\partial q^k}. \quad (6)$$

Notice that only the values $X(q_0)$ and $Y(q_0)$, and the Lie derivatives of $Y^k(\cdot)$ in the $X(q_0)$ direction are needed to compute $(\nabla_X Y)(q_0)$. Because of this, the common notation $\nabla_{\dot{q}} \dot{q}$, defined by

$$\nabla_{\dot{q}(t)} \dot{q} := (\nabla_X Y)(q(t))$$

can be used, where $X : \mathcal{G} \rightarrow T\mathcal{G}$ and $Y : \mathcal{G} \rightarrow T\mathcal{G}$ are extension fields of \dot{q} near $q(t)$ such that $X(q(t)) = \dot{q}(t)$, and $Y(q(\tau)) = \dot{q}(\tau)$ for $\tau \in (t - \epsilon, t + \epsilon)$, $\epsilon > 0$. This notation can be formalized using the concept of connection defined over maps as discussed in [3, Sec. 5.7].

Covariant derivatives of one forms can also be defined. Given a one-form, $\omega : \mathcal{G} \rightarrow T^* \mathcal{G}$, and a vector field $X : \mathcal{G} \rightarrow T\mathcal{G}$,

the covariant derivative of ω with respect to X is the one form $\nabla_X \omega$ such that, for any vector field $Y : \mathcal{G} \rightarrow T\mathcal{G}$,

$$\langle (\nabla_X \omega), Y \rangle := \mathcal{L}_X \langle \omega, Y \rangle - \langle \omega, \nabla_X Y \rangle.$$

B. Dynamics of Euler–Lagrange Mechanical Systems

The dynamics of the Euler–Lagrange mechanical system in (2), with Lagrangian defined by the kinetic energy in (1) as specified by a Riemannian metric M , can be rewritten geometrically via a special affine connection associated with M . To do this, we must first introduce the concepts of compatibility and torsion.

Given a metric M which assigns inner products $\langle \cdot, \cdot \rangle$, an affine connection ∇ is said to be *compatible with M* if, for each pair of vector fields, X, Y on \mathcal{G} and $\zeta \in T\mathcal{G}$

$$\mathcal{L}_\zeta \langle X, Y \rangle = \langle \nabla_\zeta X, Y \rangle + \langle X, \nabla_\zeta Y \rangle \quad (7)$$

where \mathcal{L}_ζ is the Lie derivative in the direction of ζ .

∇ is said to be *torsion-free* (or symmetric) if for every pair of vector fields X and Y

$$[X, Y](q) = \nabla_X Y(q) - \nabla_Y X(q) \quad (8)$$

where $[X, Y]$ is the Lie-bracket of vector fields defined by $\mathcal{L}_{[X, Y]} f := \mathcal{L}_X \mathcal{L}_Y f - \mathcal{L}_Y \mathcal{L}_X f$ for each $f : \mathcal{G} \rightarrow \mathbb{R}$. By considering X and Y to be basis vector fields, it is easy to see that the coefficients of a torsion free connection is symmetric in that $\Gamma_{ij}^k = \Gamma_{ji}^k$.

According to the Levi–Civita theorem (see, for example, [7, Th. 2.2] or [3, Th. 5.11.1]), given a Riemannian metric M on \mathcal{G} , there exists a unique affine connection, called the Levi–Civita or the Riemannian connection, which is both compatible with M and is torsion free. By applying (7) and (8) alternately, one obtains a coordinate free prescription of the Levi–Civita connection (the *Koszul formula*)

$$\begin{aligned} 2\langle \nabla_X Y, Z \rangle &= \mathcal{L}_X \langle Y, Z \rangle + \mathcal{L}_Y \langle X, Z \rangle \\ &\quad - \mathcal{L}_Z \langle X, Y \rangle - \langle X, [Y, Z] \rangle \\ &\quad + \langle Y, [Z, X] \rangle + \langle Z, [X, Y] \rangle \end{aligned} \quad (9)$$

for all vector fields X, Y, Z on \mathcal{G} . Notice, however that the Levi–Civita connection is not the only affine connection which is compatible with the metric M .

The dynamics of the Euler–Lagrange system in (2) can be written conveniently and in a coordinate free manner, as

$$M \nabla_{\dot{q}} \dot{q} = T + F_e \quad (10)$$

where

- ∇ Levi–Civita connection associated with the inertia metric M for the mechanical system in (2);
- T control force;
- F_e environment force.

Recent papers on mechanical control systems that also utilize the Levi–Civita connection description include [4], [5], and [9]. The intrinsic geometric description of the plant in (10) will be the starting point for the derivation and analysis of passive velocity field control.

C. Compatibility and Passivity

The compatibility property of an affine connection takes on a familiar form when expressed in coordinates. Let Γ_{ij}^k be the coefficients of an affine connection as in (5), and let $\mathbf{M}(\mathbf{q}) \in \mathbb{R}^{n \times n}$ be the coordinate representation of the metric M . In terms of coordinates, the compatibility condition (7) is given by:

$$\frac{\partial M_{ij}}{\partial q^k} = \Gamma_{ki}^l M_{lj} + M_{li} \Gamma_{kj}^l,$$

which implies the property that for all $\mathbf{q}(t)$ and $\dot{\mathbf{q}}(t)$, (here, \dot{q} takes the place of ζ), the matrix

$$\frac{d}{dt} \mathbf{M}(\mathbf{q}(t)) - 2\gamma(\mathbf{q}(t), \dot{\mathbf{q}}(t)) \in \mathbb{R}^{n \times n}$$

is *skew-symmetric* where

$$\gamma_{ij}(\mathbf{q}, \dot{\mathbf{q}}) := M_{ih}(\mathbf{q}) \Gamma_{kj}^h(\mathbf{q}) \dot{q}^k. \quad (11)$$

If ∇ is the Levi–Civita connection, the coefficients Γ_{ij}^k in (5) are obtained by setting $X = \partial/\partial q^j$, $Y = \partial/\partial q^i$ and $Z = \partial/\partial q^k$ in (9) so that:

$$\mathcal{C}_{kij}(\mathbf{q}) := M_{ih}(\mathbf{q}) \Gamma_{kj}^h(\mathbf{q}) = \frac{1}{2} \left(\frac{\partial M_{ji}}{\partial q^k} + \frac{\partial M_{ik}}{\partial q^j} - \frac{\partial M_{kj}}{\partial q^i} \right). \quad (12)$$

In this case, \mathcal{C}_{kij} and Γ_{ij}^k are respectively known as the Christoffel symbols of the first and second kind. Therefore, if we define the Coriolis matrix to be $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ with elements $\mathcal{C}_{ij}(\mathbf{q}, \dot{\mathbf{q}}) = \mathcal{C}_{kij}(\mathbf{q}) \dot{q}^k$, the coordinate representation of the dynamics of the mechanical control system in (10) is then given by the familiar form:

$$\mathbf{M}(\mathbf{q}) \ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}} = \mathbf{T} + \mathbf{F}_e =: \mathbf{T}_{\text{tot}}. \quad (13)$$

The compatibility of the Levi–Civita connection ∇ in (10) with the metric M then implies and recovers the so called *passivity property* familiar to the robotics community, which says that for every $\mathbf{q}(t)$ and $\dot{\mathbf{q}}(t)$, the $n \times n$ matrix

$$\dot{\mathbf{M}}(\mathbf{q}(t)) - 2\mathbf{C}(\mathbf{q}(t), \dot{\mathbf{q}}(t))$$

is *skew-symmetric* [20].

The following lemma shows that compatibility has a direct relationship to passivity and energy conservation.

Lemma 1: Let M be a metric on \mathcal{G} and ∇ an affine connection compatible with M (but not necessarily torsion-free). Suppose that the trajectories $q(t)$ of a system on \mathcal{G} are given by:

$$M \nabla_{\dot{q}} \dot{q} = T + F_e =: T_{\text{tot}}$$

where T_{tot} is the combined controlled and uncontrolled input to the system. Define the kinetic energy of the system to be $\kappa(\dot{q}) = 1/2 \langle \dot{q}, \dot{q} \rangle$. Then

- 1) the system is passive with respect to the supply rate which is the total mechanical power input

$$s(T_{\text{tot}}, \dot{q}) := \langle T_{\text{tot}}, \dot{q} \rangle; \quad (14)$$

- 2) the kinetic energy $\kappa(\dot{q}(t))$ satisfies:

$$\frac{d}{dt} \kappa(\dot{q}(t)) = \langle T_{\text{tot}}, \dot{q} \rangle;$$

3) moreover, if $T_{\text{tot}} \equiv 0$, $\kappa(\dot{q}(t))$ is a constant.

Proof: Because of the compatibility property (7) of ∇ , the kinetic energy $\kappa(\dot{q})$ satisfies

$$\begin{aligned} \frac{d}{dt}\kappa(\dot{q}(t)) &= \frac{1}{2} \frac{d}{dt} \langle \langle \dot{q}(t), \dot{q}(t) \rangle \rangle = \frac{1}{2} \mathcal{L}_{\dot{q}} \langle \langle \dot{q}, \dot{q} \rangle \rangle \\ &= \langle \langle \nabla_{\dot{q}} \dot{q}, \dot{q} \rangle \rangle = \langle T_{\text{tot}}, \dot{q} \rangle. \end{aligned} \quad (15)$$

Therefore, $\kappa(\dot{q}(t))$ remains constant if $T_{\text{tot}} \equiv 0$. Upon integration of (15), one obtains

$$\int_0^t s(T_{\text{tot}}(\tau), \dot{q}(\tau)) d\tau = \int_0^t \langle T_{\text{tot}}(\tau), \dot{q}(\tau) \rangle dt \geq -\kappa(\dot{q}(0)).$$

Since $\kappa(\dot{q}(0)) \geq 0$, the system is passive with respect to the supply rate $\langle T_{\text{tot}}, \dot{q} \rangle$. ■

Notice that in the above proof, the kinetic energy $\kappa(\dot{q})$ serves as the *storage function* for the passive system [21].

When the control torque T is determined by a control law, the closed-loop system becomes an input/output system with the environment force F_e as the input and the velocity \dot{q} as the output (Fig. 2). The passivity of this input/output system w.r.t. the supply rate in (14) depends on the specific controller. For most controllers in the literature, it is not passive. Thus, the energy of the system may increase even if no environment force is present.

D. Parallel Vector Fields and Geodesics

A vector field $V : \mathcal{G} \rightarrow T\mathcal{G}$ is said to be a *parallel vector field* of an affine connection ∇ on \mathcal{G} if for any vector $\zeta \in T\mathcal{G}$, $\nabla_{\zeta} V = 0$. A curve $c : t \mapsto c(t)$ which satisfies $\nabla_{\dot{c}} \dot{c} = 0$ is said to be a *geodesic* of ∇ . Therefore, from (10), any unforced trajectories (i.e., when $T_{\text{tot}} = 0$) of the mechanical system are *geodesics* of the Levi-Civita connection, and any geodesics $c(\cdot)$ will satisfy (10) with $T_{\text{tot}} = 0$. Also, if V is a *parallel vector field* of the Levi-Civita connection, then any integral curve of any scalar multiple of V , $q(t)$ satisfying $\dot{q}(t) = \alpha V(q(t))$ for some $\alpha \in \mathfrak{R}$ are geodesics and hence satisfies the unforced dynamics (10) with $T_{\text{tot}} = 0$. Thus, the unforced mechanical system can reproduce any flow of V , provided that the velocity of the system is correctly initialized ($\dot{q}(t_0) = \alpha V(q(t_0))$). In this sense, each parallel vector field of the Levi-Civita connection (if it exists) specifies a subset of the trajectories reproducible by the unforced mechanical system (10). Parallel vector fields are therefore means for encoding subsets of the unforced behavior of the system.

Unfortunately, parallel vector fields do not generally exist (see [3, Sec. 5.8] for a simple example on S^1). In any case, since the Levi-Civita connection is uniquely defined by the inertia metric which is a function of the plant but not of the control task, the parallel vector fields will not, in general, reflect the required task of the mechanical system anyway. As an example, consider a point mass m in \mathfrak{R}^3 , with unforced dynamics given by

$$m \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

where (x, y, z) are the Euclidean coordinates. Then any vector field given by $V(x, y, z) = a(\partial/\partial x) + b(\partial/\partial y) + c(\partial/\partial z)$ where a, b, c are constants, is a parallel vector field to the Levi-Civita connection for the inertia matrix $m\mathbf{I}$. The integral curves of these parallel vector fields are constant velocity curves (i.e., straight lines), which may or may not be the desired behavior of the mechanical control system.

E. Passivity Preserving Feedback

We describe a feedback control structure for T in (10) so that the closed-loop control system is passive with respect to the supply rate $s(F_e, \dot{q}) := \langle F_e, \dot{q} \rangle$.

A *skew-symmetric 2 tensor* \mathcal{B} on the vector space W is a bilinear map: $\mathcal{B} : W \times W \rightarrow \mathfrak{R}$ so that $\mathcal{B}(w_1, w_2) = -\mathcal{B}(w_2, w_1)$. A *two form* $\Omega : T\mathcal{G} \times T\mathcal{G} \rightarrow \mathfrak{R}$ on \mathcal{G} defines for each $q \in \mathcal{G}$ a skew-symmetric 2 tensor $\Omega(q) : T_q\mathcal{G} \times T_q\mathcal{G} \rightarrow \mathfrak{R}$. One way to construct a two-form is by *taking the wedge product* of two one-forms. Let τ, g be two one-forms. The *wedge product*, $\tau \wedge g$ is a two form given by

$$(\tau \wedge g)(v, w) = \langle \tau, v \rangle \langle g, w \rangle - \langle g, v \rangle \langle \tau, w \rangle.$$

Given a skew-symmetric 2 tensor, $\mathcal{B} : W \times W \rightarrow \mathfrak{R}$, the *contraction* of \mathcal{B} by $\zeta \in W$, denoted by $\zeta \lrcorner \mathcal{B}$ is a co-vector (i.e., $\in W^*$) so that

$$\langle \zeta \lrcorner \mathcal{B}, w \rangle = \mathcal{B}(\zeta, w), \quad \forall w \in W.$$

Similarly, the *contraction* of a two-form Ω by a vector field $V : \mathcal{G} \rightarrow T\mathcal{G}$ is a one-form, $(V \lrcorner \Omega) : T\mathcal{G} \rightarrow \mathfrak{R}$ given by

$$\langle (V \lrcorner \Omega)(q), w_q \rangle = \Omega(q)(V(q), w_q) \quad \forall w_q \in T_q\mathcal{G}, q \in \mathcal{G}.$$

Proposition 1: Let M be a metric on \mathcal{G} and ∇ an affine connection compatible with M . Suppose that the trajectories $q(t)$ of a system in \mathcal{G} are given by

$$M\nabla_{\dot{q}} \dot{q} = T + F_e.$$

Under the control law of the form

$$T(t) = \dot{q}(t) \lrcorner \Omega_t(q(t)) \quad (16)$$

where Ω_t is a possibly time varying two-form (subscript t denotes time dependence), the closed-loop system with F_e as input, and \dot{q} as output is passive with respect to the supply rate $\langle F_e, \dot{q} \rangle$.

Proof: We proceed similarly to the proof of Lemma 1. Utilizing the kinetic energy $\kappa(\dot{q})$ as the storage function, the compatibility property (7) of the connection ∇ , and the fact that $\langle T, \dot{q} \rangle = \Omega_t(\dot{q}, \dot{q}) = 0$, we obtain

$$\frac{d}{dt} \kappa(\dot{q}(t)) = \langle \langle \nabla_{\dot{q}} \dot{q}, \dot{q} \rangle \rangle = \langle F_e, \dot{q} \rangle.$$

The required result is then obtained as in Lemma 1. ■

In coordinates, two forms are represented by skew symmetric matrices. Hence, the feedback law in (16) is simply

$$\mathbf{T} = \Omega_t(\mathbf{q})\dot{\mathbf{q}}, \quad \Omega_t(\mathbf{q}) = -\Omega_t^T(\mathbf{q}) \in \mathfrak{R}^{n \times n}$$

where $\Omega_{t,ij}(\mathbf{q}) = -\Omega_{t,ji}(\mathbf{q}) = \Omega_t(q)(\partial/\partial q^i, \partial/\partial q^j)$.

F. Product Mechanical System

Let A and B be two mechanical systems with configuration spaces \mathcal{G}_A and \mathcal{G}_B and dynamics given by

$$\begin{aligned} M^A \nabla_{\dot{q}_A}^A \dot{q}_B &= T^A + F^A \\ M^B \nabla_{\dot{q}_B}^B \dot{q}_B &= T^B + F^B \end{aligned} \quad (17)$$

where

M^A and M^B	respective inertia metrics (with $\langle\langle \cdot, \cdot \rangle\rangle_A$ and $\langle\langle \cdot, \cdot \rangle\rangle_B$ as their inner products);
∇^A and ∇^B	Levi–Civita connections associated with M^A and M^B ;
$T^A \in T^*\mathcal{G}_A$	control forces;
$T^B \in T^*\mathcal{G}_B$	control forces;
$F^A \in T^*\mathcal{G}_A$	environment forces;
$F^B \in T^*\mathcal{G}_B$	environment forces;

The product system $\mathcal{P}(A, B)$ with configuration space $\mathcal{G}_A \times \mathcal{G}_B$ with $q_P = (q_A, q_B) \in \mathcal{G}_A \times \mathcal{G}_B$, such that the dynamics of q_A and q_B are described independently by (17).

Define the inertia metric M^P for the product system $\mathcal{P}(A, B)$ to be

$$\langle\langle (v_A, v_B), (w_A, w_B) \rangle\rangle_P := \langle\langle v_A, w_A \rangle\rangle_A + \langle\langle v_B, w_B \rangle\rangle_B. \quad (18)$$

Let $q_P = (q_A, q_B)$, $F_P = (F_A, F_B) \in T^*\mathcal{G}_A \times T^*\mathcal{G}_B = T^*\mathcal{G}_P$ and $T_P = (T_A, T_B) \in T^*\mathcal{G}_A \times T^*\mathcal{G}_B = T^*\mathcal{G}_P$. The dynamics of the product system given in (17) can be written as

$$M^P \nabla_{\dot{q}_P}^P \dot{q}_P = T_P + F_P. \quad (19)$$

In (19), ∇^P is the *product connection* on $\mathcal{G}_A \times \mathcal{G}_B$ defined to be

$$\nabla_{X_{AB}}^P (Y_{AB}) := (\nabla_{X_A}^A Y_A, \nabla_{X_B}^B Y_B) \quad (20)$$

where X_{AB} and Y_{AB} are vector fields on $\mathcal{G}_A \times \mathcal{G}_B$ with X_A, X_B and Y_A, Y_B being their respective projections onto $T\mathcal{G}_A$ and $T\mathcal{G}_B$.

Proposition 2: ∇^P as defined in (20) is the Levi–Civita connection on $\mathcal{G}_A \times \mathcal{G}_B$ associated with the metric M^P in (18).

Proof: Since ∇^P clearly satisfies the scaling and derivation properties, it is an affine connection. To show that ∇^P is indeed the Levi–Civita connection, it suffices to verify that it satisfies the Koszul formula (9) which uniquely specifies the Christoffel symbols for the Levi–Civita connection with the coordinate basis $\partial/\partial q^i$. ∇^P must therefore be compatible with M^P and be torsion-free. ■

Since (19) is of the same form as (10) with the metric replaced by the product metric (18) and ∇^P is compatible with M^P , from Lemma 1, the product system $\mathcal{P}(A, B)$ is passive with respect to the supply rate $s_P(T_{\text{tot}}^P, \dot{q}_P) = \langle T_{\text{tot}}^P, \dot{q}_P \rangle$ where $T_{\text{tot}}^P := T_P + F_P$. The storage function in this case can be taken to be the kinetic energy $\kappa_P(\dot{q}_P) = \langle\langle \dot{q}_P, \dot{q}_P \rangle\rangle_P$. Notice that the component mechanical systems A and B of the product system $\mathcal{P}(A, B)$ are decoupled.

The PVFC to be derived in Section IV is of the form (16) defined for a $n + 1$ dimensional product system.

IV. PASSIVE VELOCITY FIELD CONTROL

The passive velocity field control algorithm and its resulting closed-loop dynamics are derived in this section. In order to satisfy the passivity and velocity field tracking requirements, the controller structure consists of a fictitious energy storage system together with a passivity preserving coupling feedback.

A. PVFC Algorithm

Step 1: Augmented Mechanical System: In general, when $\dot{q}(t) = \alpha V(q(t))$ for some $\alpha > 0$, the kinetic energy of the plant (10) may have to increase for an interval of time, even if no environment force is present. In order to use the kinetic energy of the closed-loop system as a storage function, it is necessary to first augment the system with an energy reservoir. Thus, we define the internal dynamics of the controller to be the dynamics of a fictitious flywheel with inertia $M_f > 0$ and configuration $q^{n+1} \in S^1$, by

$$M_f \frac{d}{dt} \dot{q}^{n+1} = T_{n+1} \quad (21)$$

where T_{n+1} is the input torque to the flywheel to be determined. The flywheel dynamics (21) is a mechanical control system with inertia metric M_f and an affine connection with its coefficient given by $\Gamma_{11}^1 = 0$.

The plant (10) and the flywheel (21) form a product mechanical system which we shall refer to as the augmented mechanical system, with configuration space $\mathcal{G}_a = \mathcal{G} \times S^1 = \{q_a = (q, q^{n+1})\}$ and a Riemannian metric M^a with inner products, $\langle\langle \cdot, \cdot \rangle\rangle_a$ given by

$$\langle\langle v_a, w_a \rangle\rangle_a := \frac{1}{2} \langle\langle v, w \rangle\rangle + \frac{1}{2} M_f v^{n+1} w^{n+1}$$

for each $v_a = (v, v^{n+1})$, $w_a = (w, w^{n+1}) \in T_{q_a} \mathcal{G}_a$. The subscript and superscript a will be used to denote objects associated with the augmented system. Generally, the choice of whether to use subscript or superscript is to minimize cluttering the notations when indices are used. Thus, superscript a will be used for covectors and one forms such as F^a , and subscript a will be used for configurations such as q_a , and vectors and velocities such as V_a .

By Proposition 2, the dynamics of the augmented system are therefore

$$M^a \nabla_{\dot{q}_a}^a \dot{q}_a = T^a + F_e^a \quad (22)$$

where

∇^a	Levi–Civita connection associated with M^a ;
$T^a := (T, T_{n+1})$	augmented control input;
$F_e^a := (F_e, 0)$	augmented environment force.

The kinetic energy for the augmented system is given by:

$$\kappa_a(\dot{q}_a) := \frac{1}{2} \langle\langle \dot{q}_a, \dot{q}_a \rangle\rangle_a. \quad (23)$$

In coordinates, the dynamics of the augmented system are

$$\mathbf{M}^a(q_a) \ddot{q}_a + \mathbf{C}^a(q_a, \dot{q}_a) \dot{q}_a = \mathbf{F}_e^a + \mathbf{T}^a \quad (24)$$

where

$$\mathbf{M}^a(\mathbf{q}_a) = \begin{pmatrix} \mathbf{M}(\mathbf{q}) & \mathbf{0}_{n \times 1} \\ \mathbf{0}_{1 \times n} & M_f \end{pmatrix}$$

$$\mathbf{C}^a(\mathbf{q}_a, \dot{\mathbf{q}}_a) = \begin{pmatrix} \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) & \mathbf{0}_{n \times 1} \\ \mathbf{0}_{1 \times n} & 0 \end{pmatrix}$$

with $\mathbf{M}(\mathbf{q})$ and $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ being the inertia and Coriolis matrices in (13).

Step 2: Define an Augmented Velocity Field V_a : The rationale for letting the controller dynamics mimic that of a mechanical system (the flywheel) is to allow the kinetic energy of the augmented system to be constant when the desired velocity field is followed. This is achieved by defining a velocity field V_a on the augmented configuration space \mathcal{G}_a so that the following two conditions are satisfied:

Condition 1:

- A) **Consistency:** Let $\pi : \mathcal{G}_a \rightarrow \mathcal{G}$ be the projection operator so that if $q_a = (q, q^{n+1})$, $\pi(q_a) = q$. The augmented desired velocity field $V_a : \mathcal{G}_a \rightarrow T\mathcal{G}_a$ and the original velocity field $V : \mathcal{G} \rightarrow T\mathcal{G}$ must be π related

$$\pi_* V_a = V \quad (25)$$

where π_* denotes the push forward of π [3]. Therefore, in coordinates, $V_a(q_a)$ takes the form $\mathbf{V}_a = [\mathbf{V}^T(\mathbf{q}), V_a^{n+1}(\mathbf{q}_a)]$. This condition means that if the augmented system (22) tracks the augmented velocity field $V_a(q_a)$, the original system (10) tracks the original desired velocity field $V(q)$.

- B) **Conservation of energy:** There exists a constant $\bar{E} > 0$ so that for all $q_a \in \mathcal{G}_a$,

$$\kappa_a(V_a(q_a)) = \frac{1}{2} \langle \langle V_a(q_a), V_a(q_a) \rangle \rangle_a = \bar{E} > 0. \quad (26)$$

Given a velocity field $V : \mathcal{G} \rightarrow T\mathcal{G}$, Condition 1 can be satisfied by choosing $M_f > 0$, $V_a^i(\mathbf{q}_a) = V^i(\mathbf{q})$ for $i = 1 \dots n$ and by solving for $V_a^{n+1}(\mathbf{q}_a)$ in (26) as follows:

$$V_a^{n+1}(\mathbf{q}_a) = \sqrt{\frac{1}{M_f} [2\bar{E} - \langle \langle V(q), V(q) \rangle \rangle]}$$

after choosing a sufficiently large \bar{E} . The conservation of energy condition is specified so that power from the environment is not needed to maintain the flow of V_a .

Step 3: Coupling Control: We now define the control law for the coupling torque T^a in (22). To facilitate the presentation, define p^a and P^a to be respectively the momentum and desired momentum of the augmented system (22)

$$p^a(\dot{q}_a) := M^a \dot{q}_a, \quad (27)$$

$$P^a(q_a) := M^a V_a(q_a). \quad (28)$$

Also define

$$w^a(\dot{q}_a) := M^a \nabla_{\dot{q}_a}^a V_a. \quad (29)$$

It can be shown that $w^a(\dot{q}_a)$ is the covariant derivative of the 1-form P^a w.r.t. \dot{q}_a , i.e., $w^a(\dot{q}_a) = \nabla_{\dot{q}_a}^a P^a$. The coordinate representation of $w^a = w_i^a dq^i$, is given by

$$\mathbf{w}^a(\mathbf{q}_a, \dot{\mathbf{q}}_a) = \mathbf{M}^a(\mathbf{q}_a) \dot{\mathbf{V}}_a(\mathbf{q}_a) + \mathbf{C}^a(\mathbf{q}_a, \dot{\mathbf{q}}_a) \mathbf{V}_a(\mathbf{q}_a) \quad (30)$$

where $\mathbf{C}^a(\mathbf{q}_a, \dot{\mathbf{q}}_a)$ is the Coriolis matrix in (24).

Define the coupling control law to be

$$T^a = \dot{q}_a \underbrace{\left\{ P^a \wedge \frac{w^a(\dot{q}_a)}{2\bar{E}} \right\}}_{T^c} \underbrace{- \gamma \dot{q}_a \lrcorner [P^a \wedge p^a(\dot{q}_a)]}_{T^f} \quad (31)$$

where

- \wedge wedge product for differential forms;
- \lrcorner contraction operator;
- $\gamma \in \mathfrak{R}$ gain factor, not necessarily positive.

Notice that (31) has the form of a passivity preserving control in Lemma 1.

In coordinates, the two components of T^a are given by

$$\mathbf{T}^c(\mathbf{q}_a, \dot{\mathbf{q}}_a) = \frac{1}{2\bar{E}} \underbrace{[\mathbf{w}^a \mathbf{P}^{aT} - \mathbf{P}^a \mathbf{w}^{aT}]}_{\text{skew symmetric}} \dot{\mathbf{q}}_a$$

$$= \mathbf{G}(\mathbf{q}_a, \dot{\mathbf{q}}_a) \dot{\mathbf{q}}_a \quad (32)$$

$$\mathbf{T}^f(\mathbf{q}_a, \dot{\mathbf{q}}_a) = \gamma \underbrace{(\mathbf{P}^a \mathbf{p}^{aT} - \mathbf{p}^a \mathbf{P}^{aT})}_{\text{skew symmetric}} \dot{\mathbf{q}}_a$$

$$= \gamma \mathbf{R}(\mathbf{q}_a, \dot{\mathbf{q}}_a) \dot{\mathbf{q}}_a \quad (33)$$

where

$$\mathbf{G}(\mathbf{q}_a, \dot{\mathbf{q}}_a) = \frac{1}{2\bar{E}} [\mathbf{w}^a \mathbf{P}^{aT} - \mathbf{P}^a \mathbf{w}^{aT}] \quad (34)$$

$$\mathbf{R}(\mathbf{q}_a, \dot{\mathbf{q}}_a) = (\mathbf{P}^a \mathbf{p}^{aT} - \mathbf{p}^a \mathbf{P}^{aT}). \quad (35)$$

In (32)–(33), the arguments in \mathbf{w}^a and \mathbf{P}^a have been omitted to avoid clutter. Notice that matrices $\mathbf{G}(\mathbf{q}_a, \dot{\mathbf{q}}_a)$ and $\mathbf{R}(\mathbf{q}_a, \dot{\mathbf{q}}_a)$ are skew symmetric.

Remark 1:

- 1) In (31), T^c generates the inverse dynamics required to maintain the state of the augmented system along the desired velocity field V_a , up to a scaling determined by the current kinetic energy in the system; whereas T^f is a feedback term required for stabilization. They play similar roles to the inverse dynamics compensation and stabilization terms in the stable passivity based trajectory tracking controllers in [18], [19]. In fact, if the velocity field V_a is defined to be the reference velocity (also known as the sliding surface)

$$\mathbf{V}_a(\mathbf{q}_a, t) = \dot{\mathbf{Q}}(t) - \mathbf{K}_p(\mathbf{q}_a - \mathbf{Q}(t))$$

where \mathbf{K}_p is a positive definite gain constant and $\mathbf{Q}(t)$ is the desired trajectory to be tracked, then, the inverse dynamics compensation in [18], [19] is exactly \mathbf{w}^a in (29) except for a term to account for the time variation of the velocity field V_a . A key difference between PVFC and stable passivity based controllers in [18], [19] is that PVFC aims to achieve $\dot{q}_a \rightarrow \alpha V_a$, for an arbitrary $\alpha > 0$ which is determined by the current energy level of the system; whereas in [18],

[19], $\dot{q}_a = V_a(q_a, t)$ is desired. Because of this difference, controllers in [18] and [19] reduce, in the regulation case (i.e., when $Q(t) = Q$), to proportional-derivative control [8] incorporating both potential and kinetic energy terms in the control system. In contrast, there is no potential energy term in PVFC.

- 2) In the case when a scalar multiple of $V_a(q_a)$ is exactly tracked, i.e., $\dot{q}_a = \alpha V_a(q_a)$, the inverse dynamics term generated in T^c is given by $\alpha w_a(\dot{q}_a) = \alpha^2 w_a(V_a(q_a))$. Therefore, the definition of T^c automatically scales the inverse dynamics appropriately to the multiple that V_a is tracked. No such scaling appears in stable passivity based trajectory tracking controllers in [18] and [19].
- 3) Notice that $T^f(\alpha \dot{q}_a) = \alpha^2 T^f(\dot{q}_a)$. Hence it is quadratic in \dot{q}_a which is in contrast to most manipulator control algorithms, e.g., [18], in which the velocity feedback is linear in \dot{q}_a . This fact is responsible for the property to be presented in Section VI that robustness actually improves when the mechanical system is operating at high speed.

B. Closed-Loop Dynamics

Combining the coupling control (31) with the augmented system dynamics (22), the closed-loop dynamics become

$$M^a \nabla_{\dot{q}_a}^a \dot{q}_a - \frac{\dot{q}_a}{2E} \{P^a \wedge w^a\} + \gamma \dot{q}_a \{P^a \wedge p^a\} = F_e^a. \quad (36)$$

The coordinate representation of the closed-loop dynamics in (36) is given by

$$M^a(q_a) \ddot{q}_a + Y_\gamma(q_a, \dot{q}_a) \dot{q}_a = F_e^a \quad (37)$$

where

$$Y_\gamma(q_a, \dot{q}_a) := C^a(q_a, \dot{q}_a) - G(q_a, \dot{q}_a) - \gamma R(q_a, \dot{q}_a) \quad (38)$$

$F_e^a = [F_e, 0]^T$ are the coordinates of the environment force applied to the augmented system, and $G(q_a, \dot{q}_a)$ and $R(q_a, \dot{q}_a)$ are the skew symmetric matrices in (34)–(35).

V. GEOMETRY AND CLOSED-LOOP PROPERTIES

We now analyze the properties of the closed-loop system consisting of the augmented system (22) and the coupling control (31). The key idea in the analysis is to study the geometric property of the following *closed-loop connection*.

Define a family of affine connections on the augmented configuration space \mathcal{G}_a as follows: for each pair of vector fields X_a and Y_a on \mathcal{G}_a

$$\nabla_{X_a}^{c,\gamma} Y_a(q_a) := \nabla_{X_a}^a Y_a(q_a) - S(X_a(q_a), Y_a(q_a)) \quad (39)$$

where

$$S(v, r) := (M^a)^{-1} \left[r \left\{ P^a \wedge \left(\frac{w^a(v)}{2E} - \gamma p^a \right) \right\} \right]. \quad (40)$$

The superscript γ in $\nabla^{c,\gamma}$ is used to denote the class of closed-loop connections parameterized by the feedback gain γ .

Notice that $S(\cdot, \cdot) : T\mathcal{G} \times T\mathcal{G} \rightarrow T\mathcal{G}$ is a tensor field that takes values in $T\mathcal{G}$, i.e., it is a (1,2) type tensor field. It captures the effect of the coupling control T^a in (31). Since the sum of a

connection and any (1,2) type tensor field is a connection (see [7, Prop. 7.10] or verify the properties in (3)–(4)), $\nabla^{c,\gamma}$ is an affine connection. Using this notation, the closed-loop dynamics can be written as

$$M^a \nabla_{\dot{q}_a}^{c,\gamma} \dot{q}_a = F_e^a. \quad (41)$$

Thus, the closed-loop dynamics are of the same form as the open-loop augmented system (22) with the Levi-Civita connection ∇^a replaced by $\nabla^{c,\gamma}$, and with the input $T^a + F_e^a$ replaced by F_e^a . We shall call $\nabla^{c,\gamma}$ the closed-loop connection.

An immediate corollary to the ability to express the closed-loop dynamics in terms of an affine connection as in (41) is the following path invariance property.

Proposition 3: Consider the dynamics (41) of the closed-loop system under the control of PVFC. When the environment force is absent ($F_e^a \equiv 0$), the trajectories $q_a(t)$ are geodesics of the affine connection $\nabla^{c,\gamma}$ defined in (39). Moreover, the unforced trajectory has the following time scaling property: Suppose that $q_a(t)$ is the unforced trajectory resulting from the initial condition $\dot{q}_a(0) = v \in T_{q_a(0)} \mathcal{G}_a$. Then, if the initial condition is replaced by δv with $\delta \in \mathfrak{R}$, the unforced trajectory will be given by $q_a(\delta \cdot t)$. In particular, the path $\{q_a(t) : t \in \mathfrak{R}\}$ traced out by the closed-loop system is invariant to real scalings of the initial velocity $\dot{q}_a(0)$.

Proof: Let $n(t) = q_a(\delta t)$. Then, $\dot{n}(t) = \delta \dot{q}_a(\delta t)$ and the initial velocities are scaled by $\dot{n}(0) = \delta \dot{q}_a(0)$. Assuming $F_e^a \equiv 0$ and applying the scaling property of affine connections (3)–(4), we have

$$M^a \nabla_{\dot{n}}^{c,\gamma} \dot{n} = \delta^2 M^a \nabla_{\dot{q}_a(\delta t)}^{c,\gamma} \dot{q}_a(\delta t) = F_e^a \equiv 0.$$

This shows that $n(t)$ satisfies (41) and $q_a(\delta t)$ is the unforced closed-loop trajectory when the initial velocity is scaled by δ . ■

Proposition 3 demonstrates that the PVFC encodes the task motion in such a way that it specifies the path that the unforced closed-loop trajectories will trace out, but allows the traversal speed to be determined by the initial speed.

We now give the properties of the *closed-loop connection*, $\nabla^{c,\gamma}$. First notice that $S(\zeta_a, X)$ can be expanded using (27)–(29) as follows:

$$S(\zeta_a, X) = \frac{\langle \langle V_a, X \rangle \rangle_a}{2E} \nabla_{\zeta_a}^a V_a - V_a \frac{\langle \langle \nabla_{\zeta_a}^a V_a, X \rangle \rangle_a}{2E} - \gamma (\zeta_a \langle \langle V_a, X \rangle \rangle_a - V_a \langle \langle \zeta_a, X \rangle \rangle_a). \quad (42)$$

From the above, it can be easily verified that

$$\langle \langle S(\zeta_a, X), U \rangle \rangle = -\langle \langle X, S(\zeta_a, U) \rangle \rangle. \quad (43)$$

This is expected since $S(v, r)$ as defined in (40) is a contraction of a two-form.

Theorem 1: The closed-loop connection $\nabla^{c,\gamma}$ is compatible with the Riemannian metric M^a i.e., for any X, U vector fields on \mathcal{G}_a and $\zeta_a \in T\mathcal{G}_a$,

$$\mathcal{L}_{\zeta_a} \langle \langle X, U \rangle \rangle_a = \langle \langle \nabla_{\zeta_a}^{c,\gamma} X, U \rangle \rangle_a + \langle \langle X, \nabla_{\zeta_a}^{c,\gamma} U \rangle \rangle_a$$

where \mathcal{L}_{ζ_a} denotes the directional (Lie) derivative in the direction of ζ_a .

Proof: Using the definition of $S(\zeta, X)$ in (40), we have

$$\nabla_{\zeta_a}^a X = \nabla_{\zeta_a}^{c,\gamma} X - S(\zeta_a, X).$$

Since ∇^a is compatible with the metric, we obtain

$$\mathcal{L}_{\zeta_a} \langle \langle X, U \rangle \rangle_a = \langle \langle \nabla_{\zeta_a}^a X, U \rangle \rangle_a + \langle \langle X, \nabla_{\zeta_a}^a U \rangle \rangle_a.$$

These imply that

$$\begin{aligned} \mathcal{L}_{\zeta_a} \langle \langle X, U \rangle \rangle_a &= \langle \langle \nabla_{\zeta_a}^{c,\gamma} X, U \rangle \rangle_a + \langle \langle X, \nabla_{\zeta_a}^{c,\gamma} U \rangle \rangle_a \\ &\quad - \langle \langle S(\zeta_a, X), U \rangle \rangle_a - \langle \langle X, S(\zeta_a, U) \rangle \rangle_a. \end{aligned}$$

Therefore $\nabla^{c,\gamma}$ is compatible with M^a iff

$$\langle \langle S(\zeta_a, X), U \rangle \rangle = -\langle \langle X, S(\zeta_a, U) \rangle \rangle$$

which, from (43) is indeed the case. ■

Notice that entries of $\mathbf{Y}(\mathbf{q}_a, \dot{\mathbf{q}}_a)$ in (37) are equivalent to γ_{ij} in (11) for $\nabla^{c,\gamma}$. Thus, from the discussion in Section III-C, the compatibility of $\nabla^{c,\gamma}$ with M^a can also be seen from the property that the matrix

$$\dot{M}^a(\mathbf{q}_a) - 2\mathbf{Y}(\mathbf{q}_a, \dot{\mathbf{q}}_a)$$

is skew symmetric. The passivity property of the closed-loop system is an immediate consequence of this fact.

Corollary 1: Consider the closed-loop feedback system (41) with the environment force F_e as input and the velocity \dot{q} as output. The kinetic energy of the augmented system κ_a in (23) satisfies

$$\frac{d}{dt} \kappa_a(\dot{q}(t)) = \langle F_e, \dot{q} \rangle. \quad (44)$$

Consequently, the system is passive with respect to the supply rate $s(F_e, \dot{q}) := \langle F_e, \dot{q} \rangle$ which is the power input from the environment.

Proof: Recall that the augmented environment force F_e^a in (41) is given by $F_e^a = (F_e, 0)$. Thus, $\langle F_e^a, \dot{q}_a \rangle = \langle F_e, \dot{q} \rangle$. Since $\nabla^{c,\gamma}$ is compatible with M^a (Theorem 1), the required results are direct consequences of Lemma 1. ■

Notice that since ∇^a in (22) is the Levi-Civita connection associated with the inertia M^a , it is the unique connection that is both compatible with M^a and is torsion free. Since the closed-loop connection $\nabla^{c,\gamma}$ is also compatible with M^a , it cannot be torsion free if it differs from ∇^a . Thus, by giving up the torsion free property, the geodesics of $\nabla^{c,\gamma}$, which are the unforced closed-loop trajectories of (41), have some desirable properties. We now characterize these unforced closed-loop trajectories.

Let us decompose $\nabla^{c,\gamma}$ into two components

$$\nabla_{\zeta_a}^{c,\gamma} X_a = \nabla_{\zeta_a}^{c,0} X_a + \gamma(M^a)^{-1} \{X_a\} (P^a \wedge p^a(\zeta_a)). \quad (45)$$

Thus, $\nabla^{c,0}$ is the closed-loop connection when the feedback gain $\gamma = 0$.

Theorem 2:

- 1) The augmented desired velocity field V_a satisfying Condition 1, is a parallel vector field of the connection $\nabla^{c,0}$ i.e.,

$$\nabla_{\zeta_a}^{c,0} V_a = 0, \quad \forall \zeta_a \in T\mathcal{G}_a. \quad (46)$$

- 2) For any feedback gain $\gamma \in \mathfrak{R}$ in (39), we have $\nabla_{V_a}^{c,\gamma} V_a = 0$. Therefore, for any scalar $\alpha \in \mathfrak{R}$, the flow of αV_a is a geodesic of $\nabla^{c,\gamma}$.

Proof: When $\gamma = 0$, we have

$$\nabla_{\zeta_a}^{c,0} X = \nabla_{\zeta_a}^a X - \frac{\langle \langle V_a, X \rangle \rangle_a}{2\bar{E}} \nabla_{\zeta_a}^a V_a + V_a \frac{\langle \langle \nabla_{\zeta_a}^a V_a, X \rangle \rangle_a}{2\bar{E}}.$$

Since $\langle \langle V_a, V_a \rangle \rangle_a = 2\bar{E}$, a constant by design in (26), and $\mathcal{L}_{\zeta_a} \langle \langle V_a, V_a \rangle \rangle_a = 2\langle \langle \nabla_{\zeta_a}^a V_a, V_a \rangle \rangle_a$ because ∇^a is compatible with M^a , $\langle \langle \nabla_{\zeta_a}^a V_a, V_a \rangle \rangle_a = 0$. Thus

$$\begin{aligned} \nabla_{\zeta_a}^{c,0} V_a &= \nabla_{\zeta_a}^a V_a - \frac{\langle \langle V_a, V_a \rangle \rangle_a}{2\bar{E}} \nabla_{\zeta_a}^a V_a + V_a \frac{\langle \langle \nabla_{\zeta_a}^a V_a, V_a \rangle \rangle_a}{2\bar{E}} \\ &= 0. \end{aligned}$$

Hence V_a is a parallel vector field with respect to the closed-loop connection, $\nabla^{c,0}$, i.e., $\nabla^{c,\gamma}$ with $\gamma = 0$.

Since V_a is a parallel vector field of $\nabla^{c,\gamma}$ when $\gamma = 0$, to see that the flow of V_a are geodesics of $\nabla^{c,\gamma}$, for $\gamma \neq 0$, i.e., $\nabla_{V_a}^{c,\gamma} V_a = 0$, we need only verify that the second term in (45) vanishes when $X_a = \zeta_a = V_a$. This is indeed the case since by taking the inner product of this term with $*$ in $T\mathcal{G}_a$, we have

$$\begin{aligned} [X_a](P^a \wedge p^a(\zeta_a))[*] &= ((P^a \wedge p^a(\zeta_a))(X_a, *) \\ &= \langle \langle V_a, X_a \rangle \rangle_a \langle \langle \zeta_a, * \rangle \rangle_a \\ &\quad - \langle \langle \zeta_a, X_a \rangle \rangle_a \langle \langle V_a, * \rangle \rangle_a \end{aligned}$$

which vanishes at $X_a = \zeta_a = V_a$. ■

Theorem 2 can be comprehended as follows: from the discussion in Section III, since V_a is a parallel vector field of $\nabla^{c,0}$ (the stabilizing term T^f in (31) is absent), the flow of αV_a for any $\alpha \in \mathfrak{R}$ are feasible closed-loop trajectories of the closed-loop system (41). The second item of Theorem 2 shows that this is also true for any feedback gain γ . Indeed T^f in (31) vanishes when $\dot{q}_a = \alpha V_a$, for any α . What remains to be shown is that these flows are in fact stable solutions.

For any $\alpha \in \mathfrak{R}$, define the α -velocity error as

$$e_\alpha(t) := \dot{q}_a(t) - \alpha V_a(q_a(t)). \quad (47)$$

Thus, if $e_\alpha(t) = 0$, $\forall t$ for some α , the velocity field tracking objective is satisfied.

The closed-loop dynamics (41) can be written as

$$M^a \nabla_{\dot{q}_a}^{c,0} \dot{q}_a = T^f + F_e^a \quad (48)$$

where T^f is the second (stabilizing) term in the coupling control in (33).

Proposition 4: For any $\alpha \in \mathfrak{R}$, consider $T^f + F_e^a$ as the input to the system, and e_α in (47) as its output. Then

$$\frac{d}{dt} \langle \langle e_\alpha, e_\alpha \rangle \rangle_a = \langle T^f + F_e^a, e_\alpha \rangle.$$

Thus, the closed-loop system is passive with respect to the supply rate $\langle T^f + F_e^a, e_\alpha \rangle$.

Proof: Subtracting $\alpha M^a \nabla_{\dot{q}_a}^{c,0} V_a = 0$ from (48), we obtain

$$M^a \nabla_{\dot{q}_a}^{c,0} e_\alpha = T^f + F_e^a.$$

The required result follows from a proof similar to that of Corollary 1 making use of the fact that $\nabla^{c,0}$ is compatible with M^a .

■

Proposition 5: For the closed-loop system (41)

$$\begin{aligned} \frac{d}{dt} \langle \langle V_a(q(t)), \dot{q}_a(t) \rangle \rangle_a &= \gamma [\langle \langle \dot{q}_a, \dot{q}_a \rangle \rangle_a \langle \langle V_a, V_a \rangle \rangle_a \\ &\quad - \langle \langle V_a, \dot{q}_a \rangle \rangle_a^2] + \langle F_e^a, V_a \rangle. \end{aligned} \quad (49)$$

In particular, when $F_e \equiv 0$, $\langle \langle V_a(q(t)), \dot{q}_a(t) \rangle \rangle_a$ is a nondecreasing function of time t if $\gamma > 0$; a nonincreasing function of time if $\gamma < 0$; and a constant if $\gamma = 0$.

Proof: Since $\nabla^{c,0}$ is compatible with M^a (Theorem 1), we have

$$\mathcal{L}_{\dot{q}_a} \langle \langle V_a, \dot{q}_a \rangle \rangle_a = \langle \langle V_a, \nabla_{\dot{q}_a}^{c,0} \dot{q}_a \rangle \rangle_a + \langle \langle \nabla_{\dot{q}_a}^{c,0} V_a, \dot{q}_a \rangle \rangle_a.$$

Further, because V_a is a parallel vector field (Theorem 2), we have

$$\mathcal{L}_{\dot{q}_a} \langle \langle V_a, \dot{q}_a \rangle \rangle_a = \langle \langle V_a, \nabla_{\dot{q}_a}^{c,0} \dot{q}_a \rangle \rangle_a.$$

Now

$$M^a \nabla_{\dot{q}_a}^{c,\gamma} \dot{q}_a = M^a \nabla_{\dot{q}_a}^{c,0} \dot{q}_a + \gamma \{ \dot{q}_a \} (P^a \wedge p^a)$$

whereas from (41)

$$M^a \nabla_{\dot{q}_a}^{c,\gamma} \dot{q}_a = F_e^a.$$

Hence

$$\begin{aligned} \frac{d}{dt} \langle \langle V_a(q(t)), \dot{q}_a(t) \rangle \rangle_a &= \mathcal{L}_{\dot{q}_a} \langle \langle V_a, \dot{q}_a \rangle \rangle_a = \langle \langle V_a, \nabla_{\dot{q}_a}^{c,0} \dot{q}_a \rangle \rangle_a \\ &= \langle V_a \rangle (F_e^a - \gamma \{ \dot{q}_a \} (P^a \wedge p^a)) \\ &= \gamma [\langle \langle \dot{q}_a, \dot{q}_a \rangle \rangle_a \langle \langle V_a, V_a \rangle \rangle_a \\ &\quad - \langle \langle V_a, \dot{q}_a \rangle \rangle_a^2] + \langle F_e^a, V_a \rangle. \end{aligned}$$

By the Schwartz inequality, the expression between the square parenthesis is nonnegative. Thus, as required, when $F_e^a \equiv 0$, $d/dt \langle \langle V_a(q(t)), \dot{q}_a(t) \rangle \rangle_a$ and γ take on the same signs. ■

Define the angle θ between \dot{q}_a and $V_a(q_a)$ by

$$\cos \theta := \frac{\langle \langle V_a, \dot{q}_a \rangle \rangle_a}{\sqrt{\langle \langle V_a, V_a \rangle \rangle_a \cdot \langle \langle \dot{q}_a, \dot{q}_a \rangle \rangle_a}}, \quad \theta \in [-\pi, \pi].$$

Because V_a is constructed to satisfy the energy conservation condition in Condition (1), $\langle \langle V_a, V_a \rangle \rangle_a = 2\bar{E}$ is a constant. Moreover, if $F_e \equiv 0$, $\langle \langle \dot{q}_a, \dot{q}_a \rangle \rangle_a$ is the kinetic energy, which is also a constant by Corollary 1. Thus, $|\theta|$ increases while $\langle \langle V_a, \dot{q}_a \rangle \rangle_a$ decreases, and vice versa. Therefore, Proposition 5 says that the magnitude of the angle decreases if $\gamma > 0$, increases if $\gamma < 0$ and remains constant when $\gamma = 0$. This property gives rise to Theorem 3 which concerns the stability and convergence properties of the closed-loop system.

We are now able to state our main stability results. Define $\beta \in \mathfrak{R}$ up to its sign, so that β^2 is the ratio between the kinetic energy $\kappa_a(\dot{q}_a)$ in (23) and the constant \bar{E} which was defined in (26)

$$\beta^2(t) = \frac{\kappa_a(\dot{q}_a(t))}{\bar{E}}.$$

Theorem 3: The closed-loop system (41) consisting of (Fig. 2) the augmented system (22) and the coupling control law (27)–(29) and (31) has the following properties:

- 1) for any $\alpha \in \mathfrak{R}$, let $e_\alpha := \dot{q}_a - \alpha V_a(q_a)$. Suppose that $\gamma\alpha \geq 0$, then, in the absence of environment forces ($F_e \equiv 0$), $e_\alpha = 0$ is a Lyapunov stable solution;
- 2) suppose that no environment force ($F_e \equiv 0$) is present (hence $\beta^2 = \kappa_a/\bar{E}$ is a constant). The solution $e_\beta = 0$ or $\dot{q}_a - \beta V_a(q_a) = 0$ is globally exponentially stable (unstable), except from a set of measure 0 if $\gamma\beta > 0$ (if $\gamma\beta < 0$). The rate of convergence (divergence) in a neighborhood of $e_\beta = 0$ is given by $2\gamma\beta\bar{E}$.

Proof: Let $\alpha \in \mathfrak{R}$. Notice that

$$\langle \langle e_\alpha, e_\alpha \rangle \rangle_a = \langle \langle \dot{q}_a, \dot{q}_a \rangle \rangle_a + \alpha^2 \langle \langle V_a, V_a \rangle \rangle_a - 2\alpha \langle \langle V_a, \dot{q}_a \rangle \rangle_a \quad (50)$$

where e_α is the α -velocity error defined in (47). Notice that $\langle \langle V_a, V_a \rangle \rangle_a = 2\bar{E}$ by design in (26) and $\langle \langle \dot{q}_a(t), \dot{q}_a(t) \rangle \rangle_a$ is constant when $F_e \equiv 0$ (Corollary 1). Thus, Proposition 5 implies that $\langle \langle e_\alpha(t), e_\alpha(t) \rangle \rangle_a$ is nonincreasing when $F_e \equiv 0$ and $\alpha\gamma \geq 0$. Therefore, given $\epsilon > 0$, for all $e_\alpha(0)$ s.t. $\langle \langle e_\alpha(0), e_\alpha(0) \rangle \rangle_a < \epsilon$, $\langle \langle e_\alpha(t), e_\alpha(t) \rangle \rangle_a < \epsilon$ for all $t \geq 0$. This shows that $e_\alpha = 0$ is Lyapunov stable when $\alpha\gamma \geq 0$.

Consider now $\beta \in \mathfrak{R}$ satisfying i) $2\bar{E}\beta^2 = \langle \langle \dot{q}_a, \dot{q}_a \rangle \rangle_a$, and ii) $\beta \cdot \gamma > 0$. Corollary 1 shows that β is constant if $F_e = 0$.

Define $W_\beta(t) := 1/2 \langle \langle e_\beta(t), e_\beta(t) \rangle \rangle_a$. Setting $\alpha = \beta$ in (50), we obtain

$$\frac{W_\beta}{\beta} = 2\beta\bar{E} - \langle \langle V_a, \dot{q}_a \rangle \rangle_a. \quad (51)$$

Assuming $F_e \equiv 0$, the time derivative of W_β is given by the following, after making use of (51) and (49), and using the identity $(a^2 - b^2) = (a+b)(a-b)$:

$$\dot{W}_\beta = -\gamma \cdot \beta (2\beta\bar{E} - \langle \langle V_a, \dot{q}_a \rangle \rangle_a) (2\beta\bar{E} + \langle \langle V_a, \dot{q}_a \rangle \rangle_a).$$

Thus

$$\dot{W}_\beta = -2\gamma\beta(2\bar{E})\mu(\dot{q}_a)W_\beta \quad (52)$$

where

$$\mu(\dot{q}_a) := \frac{1}{2} \left(1 + \frac{\langle \langle V_a, \dot{q}_a \rangle \rangle_a}{2\beta\bar{E}} \right).$$

By the Schwartz inequality, $|\langle \langle V_a, \dot{q}_a \rangle \rangle_a| \leq 2|\beta|\bar{E}$. Hence $\mu(\dot{q}_a) \geq 0$ and consequently the right-hand side of (52) is nonpositive. Moreover, since $\beta\gamma > 0$, Proposition 5 shows that $\mu(\dot{q}_a(t))$ is nondecreasing in time.

Notice that $8\beta^2\bar{E}\mu(\dot{q}_a) + \langle \langle e_\beta, e_\beta \rangle \rangle_a = 8\beta^2\bar{E}$. Therefore, for any $\epsilon \in [0, 8\beta^2\bar{E}]$, we have, whenever $\langle \langle e_\beta, e_\beta \rangle \rangle_a \leq \epsilon$

$$\mu(\dot{q}_a) \geq 1 - \frac{\epsilon}{8\beta^2\bar{E}}$$

Thus, for all $\dot{q}_a(0)$ such that $\langle \langle e_\beta(0), e_\beta(0) \rangle \rangle_a < \epsilon$

$$\dot{W}_\beta(t) \leq - \left[4\bar{E}\gamma\beta \left(1 - \frac{\epsilon}{8\beta^2\bar{E}} \right) \right] W_\beta(t) \quad \forall t \geq 0.$$

Hence, if $\langle\langle e_\beta(0), e_\beta(0) \rangle\rangle_a < \epsilon$, $W_\beta(t) \rightarrow 0$ exponentially at a rate given by

$$\left[4\bar{E}\gamma\beta \left(1 - \frac{\epsilon}{8\beta^2\bar{E}} \right) \right].$$

Since ϵ can be made arbitrarily close to $8\beta^2\bar{E}$ (this is the maximum that $\langle\langle e_\beta(0), e_\beta(0) \rangle\rangle_a$ can achieve), exponential convergence is global (although the bound on the exponential rate becomes smaller as $\epsilon \rightarrow 8\beta^2\bar{E}$), except from the set of measure zero characterized by $\langle\langle e_\beta(0), e_\beta(0) \rangle\rangle_a = 8\beta^2\bar{E}$. As $\epsilon \rightarrow 0$, the convergence rate for W_β approaches $4\beta\gamma\bar{E}$ in the neighborhood of $e_\beta = 0$, and thus $e_\beta \rightarrow 0$ at a rate that approaches $2\beta\gamma\bar{E}$ in the neighborhood of $e_\beta = 0$.

The same argument shows that $W_{-\beta} \rightarrow 0$ exponentially as $t \rightarrow -\infty$. Hence, $e_{-\beta} = 0$ is exponentially unstable. ■

Theorem 3 shows that the control stabilizes a whole family of velocity fields which are scalar multiples of V_a as long as the scaling has the same sign as the feedback gain γ . Thus, γ can be used to choose the sense in which the desired velocity field should be followed. When environment forces are absent, the velocity \dot{q}_a exponentially converges to the particular multiple of V_a consistent with the amount of energy in the system. By injecting or extracting energy (through additional control loops via F_e), the speed at which the velocity field is followed can be altered.

VI. ROBUSTNESS

In most applications, the mechanical system will encounter environment forces, e.g., friction. In addition, the effects of model parameter uncertainties can also be considered as environment forces. The effect of these forces on the ability of the system to track a multiple of the given velocity field is analyzed in this section. The following theorem characterizes the robustness of the passive velocity field controller to disturbances “parallel” and “normal” to the desired momentum, and to disturbances that alter the total energy in the robot system.

We shall denote the norms on $T\mathcal{G}_a$ and $T^*\mathcal{G}_a$ by

$$\begin{aligned} \|r\|_a &:= \langle\langle r, r \rangle\rangle_a^{1/2} \quad \forall r \in T_{q_a}\mathcal{G}_a \quad \forall q_a \in \mathcal{G}_a \\ \|F\|_a &:= \langle\langle (M^a)^{-1}F, (M^a)^{-1}F \rangle\rangle_a^{1/2} \quad \forall F \in T^*\mathcal{G}_a \\ &\quad \forall q_a \in \mathcal{G}_a. \end{aligned}$$

Since these two norm functions operate on dual spaces ($T\mathcal{G}_a$ and $T^*\mathcal{G}_a$), there should not be any confusion.

Theorem 4: Consider the closed-loop dynamics given by (41). Let

$$F_e^a(t) = \delta(t)P^a(q_a(t)) + F^\perp(t)$$

be the orthogonal decomposition of the environment force in (22) with respect to the metric M^a , such that $\langle F^\perp, V_a(q_a) \rangle = 0$ and $P^a = M^a V_a$ is the desired momentum in (28).¹ Assume that for some $a, b, c_1, c_2 > 0$

- i) $\delta(t) > -a\beta(t)^2$;
- ii) $(d/dt)\beta^2(t) > -2b|\beta(t)|^3$;
- iii) $\|F^\perp(t)\|_a \leq \sqrt{2\bar{E}} \min(c_2|\beta(t)|, c_1\beta^2(t))$;

¹Notice that $\delta(t)$ has the unit of sec^{-1} .

where

$$\beta(t) = \text{sign}(\gamma) \left(\frac{\kappa_a(\dot{q}_a(t))}{\bar{E}} \right)^{1/2}.$$

Let $e_\beta(t) = \dot{q}_a - \beta(t)V_a(q_a(t))$. Then

- 1) suppose $c_1 = c_2 \equiv 0$ (i.e., environment force in the normal direction, $F^\perp \equiv 0$) and the environment power input $\langle F_e, \dot{q} \rangle$, is finite. Given any $G_0 \in (0, 1)$, there exists $\gamma_1(G_0)$, s.t. if the feedback gain is chosen with $\gamma > \gamma_1(G_0)$, then for any initial error $e_\beta(0)$ bounded by $\|e_\beta(0)\|_a^2 < 8\bar{E}\beta^2(0)G_0$, $e_\beta(t) \rightarrow 0$;
- 2) in general, given any $G_0 \in (0, 1)$, and $\epsilon > 0$, there exists $\gamma_2(G_0, \epsilon)$, so that: if $\gamma > \gamma_2(G_0, \epsilon)$ and $\|e_\beta(0)\|_a^2 < 8\bar{E}\beta^2(0)G_0$, then, there exists $T > 0$ such that $\|e_\beta(t)\|_a < \epsilon$, $\forall t > T$. Moreover, if $\|e_\beta(0)\|_a \leq \epsilon$, $\|e_\beta(t)\|_a \leq \epsilon$, $\forall t \geq 0$.

The bounds on the gains γ_1 and γ_2 are given by

$$\begin{aligned} \gamma_1 &= \frac{1}{2\bar{E}(1-G_0)} \left(\frac{c_1}{2G_0^{1/2}} + a + b + c_1 + \lambda \right); \\ \gamma_2 &= \max \left[\gamma_1, \frac{1}{2\bar{E}(1-G_0)} \left(\frac{c_2\sqrt{2\bar{E}}}{\epsilon} + a + c_1 + \lambda \right) \right]. \end{aligned}$$

for some $\lambda > 0$.

The proof of this theorem is quite tedious and is included in Appendix A. The basic idea is to decompose the augmented environment force F_e^a into a component in the direction of the desired momentum $P^a = M^a V_a$ and its orthogonal complement. Then, the effect of each component on the derivative of the Lyapunov function $\langle\langle e_\beta(t), e_\beta(t) \rangle\rangle_a/2$ is investigated and bounded.

Remark 2:

- 1) Assumption i) specifies an upper bound on the disturbance in the negative direction of the desired momentum P^a . Notice that Assumption i) does not impose any constraints on the magnitude of this component if $\delta(t)$ is positive. Assumption ii) specifies the maximum rate of energy dissipation. Assumption iii) specifies the bound on the disturbance normal to P^a .
- 2) Conclusion 1. states that $\dot{q}_a(t) \rightarrow \beta(t)V_a(q_a(t))$ as long as the environment forces or disturbances are only in the direction of the desired momentum, and if the stabilizing gain γ is sufficiently high. This result is in contrast to passivity based trajectory controllers [18], [19] which only guarantee that the tracking error converges to 0 in the absence of any disturbances.
- 3) Conclusion 2. states that, given a sufficiently high stabilizing gain γ , $e_\beta(t)$ is ultimately bounded by an arbitrarily small bound for appropriately bounded environment forces.
- 4) The constraints imposed by Assumptions i)–iii) all become less restrictive when the kinetic energy in the system is larger. Notice that Assumptions i) and ii) will be simultaneously satisfied if $\|F_e^a\|_a \leq \sqrt{2\bar{E}} \min(a, b)\beta^2$, and assumptions i), ii), and iii) are satisfied if $\|F_e^a\|_a \leq \min(a\beta^2, b\beta^2, c_1\beta^2, c_2|\beta|)\sqrt{2\bar{E}}$. Since β^2 is proportional to the kinetic energy in the system, the controller’s ability to withstand disturbances improves as

the amount of energy in the system ($\beta^2 \bar{E}$) increases, i.e., when the system is moving at higher speed. This property is due to the fact that the stabilizing term T^f in (33) is quadratic with \dot{q}_a . This result is not true for controllers in the literature that make use of linear velocity feedback, e.g., [18] and [19].

- 5) The effect of model parameter errors (e.g., uncertainty in inertia parameters of the system) can be modeled as environment forces. These effects will grow in a quadratic manner with increasing \dot{q}_a . Since the robustness property of a system controlled by PVFC also improves when the kinetic energy of the system increases, Theorem 4 predicts that the performance of the control system should not degrade as the operating speed increases. This prediction has been confirmed by experimental results in the companion paper [15].
- 6) Notice that in assumptions i) and ii), no bounds are needed on the disturbances in the positive direction of the desired momentum P^a or on the rate of power input. Thus, disturbances that push in the positive direction of P^a or provide positive power do not degrade the system's ability to track the scaling of the desired velocity field. The fact that disturbances in the P^a direction are effectively inconsequential can be attributed to the control requirement that only $e_\alpha = \dot{q}_a - \alpha V_a$ matters for an arbitrary α . Thus, disturbances in the P^a direction merely change the scaling α .
- 7) The asymptotic effects of disturbances that act in the direction of the desired momentum can be eliminated by a sufficiently high gain. As already noted, disturbances in the positive direction of P^a do not affect the tracking performance.
- 8) If the amount of energy level in the system (i.e., β) is desired to vary, Theorem 4 suggests that external forcing in the P^a direction would minimize the adverse effect on the velocity field tracking performance, as long as the energy dissipation rate is not too high. This is utilized in [15] to synthesis an additional control loop to regulate the nominal speed of operation in the contour following experiments.
- 9) Theorem 4 also characterizes the environment forces that tend to affect performance adversely. They have the following properties:
 - a) they have large components orthogonal to the desired momentum;
 - b) they cause dissipation of kinetic energy;
 - c) the component parallel to the desired momentum acts in the opposite direction to the desired momentum.

VII. CONCLUSION

In this paper, we develop and analyze a new control methodology for fully actuated mechanical systems. Two key underlying concepts are: 1) control tasks are represented in terms of velocity fields; 2) the closed-loop input/output system is passive with the environment force as input, the system velocity as output, and the environment mechanical power as the supply rate. The control scheme consists of the dynamics of a fictitious energy storage element (like a flywheel) together with a coupling force which conservatively transfers energy between the different components in the augmented system. The resulting closed-loop system dynamics can be described using a closed-loop affine connection. Compared

with the open-loop mechanical system whose dynamics can be described by the Levi–Civita connection, the closed-loop connection, although not Levi–Civita, is also compatible with the augmented inertia metric. However, the flow of any scaled multiple of the desired velocity field is necessarily a geodesic of the closed-loop connection, thus allowing the closed-loop system to perform useful tasks. The velocity of the system converges exponentially to the scaled multiple of the desired velocity field. This scaling is determined by the energy in the system and corresponds to the speed at which the task is executed. Robustness properties of the closed-loop system to environment forces exhibit some strong directional and energy dependence. The closed-loop system is very effective in tracking a multiple of the desired velocity field and in counteracting the detrimental effect of environment disturbances when the disturbance is in the direction of the desired momentum of the system. Performance also improves when the system is moving at high speed. In the companion paper [15], a contour following problem is formulated as a passive velocity field control problem and the control strategy proposed in this paper is applied. Experimental results verifying the properties of the control scheme will also be presented. Recently (after this paper was first submitted), an adaptive version of PVFC has also been developed that alleviates the need for precise knowledge of the inertia parameters of the mechanical system [10].

APPENDIX

PROOF OF THE ROBUSTNESS RESULT IN THEOREM 4

The following result will be useful to prove the robustness results in Theorem 4.

Lemma 2: Let $F_e^a(t) = \delta(t)P^a + F^\perp(t)$ be the orthogonal decomposition of F_e^a so that $\delta(t) \in \mathfrak{R}$ and $\langle F^\perp(t), V_a \rangle = 0$. The time derivative of

$$W_{\beta(t)}(t) = \frac{1}{2} \langle \langle e_{\beta(t)}(t), e_{\beta(t)}(t) \rangle \rangle_a$$

satisfies

$$\begin{aligned} \frac{d}{dt} W_\beta \leq & - \left[4\bar{E}\gamma\beta\mu(\dot{q}_a) - \frac{2}{\beta} \left(\frac{\|F^\perp\|_a}{\sqrt{2\bar{E}}} - \min(0, \delta) \right) \right] \\ & \times W_\beta + \langle F^\perp, e_\beta \rangle \end{aligned} \quad (53)$$

where $\mu(\dot{q}_a) = \frac{1}{2}(1 + \langle \langle V_a, \dot{q}_a \rangle \rangle_a / 2\beta\bar{E})$.

Proof: (Lemma 2) Using (49),(44),(51), the effect of F_e^a on W_β is given by

$$\frac{d}{dt} W_\beta = -4\gamma\beta\bar{E}\mu(\dot{q}_a)W_\beta + \langle F_e^a, e_\beta \rangle - \frac{\langle F_e, \dot{q} \rangle \langle \langle V_a, e_\beta \rangle \rangle_a}{2\beta\bar{E}}. \quad (54)$$

Since F_e^a enters linearly, we can treat the orthogonal components of F_e^a separately. For the δP^a component

$$\begin{aligned} A := & \langle \delta P^a, e_\beta \rangle - \frac{\langle \delta P^a, \dot{q}_a \rangle \langle \langle V_a, e_\beta \rangle \rangle_a}{2\beta\bar{E}} \\ = & \delta \langle \langle V_a, e_\beta \rangle \rangle - \delta\beta \frac{\langle \langle V_a, V_a \rangle \rangle_a \langle \langle V_a, e_\beta \rangle \rangle_a}{2\beta\bar{E}} \\ & - \delta \frac{\langle \langle V_a, e_\beta \rangle \rangle_a \langle \langle V_a, e_\beta \rangle \rangle_a}{2\beta\bar{E}} \\ = & -\delta \frac{\langle \langle V_a, e_\beta \rangle \rangle_a \langle \langle V_a, e_\beta \rangle \rangle_a}{2\beta\bar{E}} \leq -\frac{2}{\beta} \min(0, \delta) W_\beta. \end{aligned}$$

For the complementary component

$$\begin{aligned}
A^\perp &:= \langle F^\perp, e_\beta \rangle - \frac{\langle F^\perp, \dot{q}_a \rangle \langle V_a, e_\beta \rangle_a}{2\beta\bar{E}} \\
&= \langle F^\perp, e_\beta \rangle - \beta \frac{\langle F^\perp, V_a \rangle \langle V_a, e_\beta \rangle_a}{2\beta\bar{E}} \\
&\quad - \frac{\langle F^\perp, e_\beta \rangle \langle V_a, e_\beta \rangle_a}{2\beta\bar{E}} \\
&= \langle F^\perp, e_\beta \rangle - \frac{\langle F^\perp, e_\beta \rangle \langle V_a, e_\beta \rangle_a}{2\beta\bar{E}} \\
&\leq \langle F^\perp, e_\beta \rangle + \frac{2\|F^\perp\|_a}{\sqrt{2\bar{E}}\beta} W_\beta.
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{d}{dt} W_\beta &= -4\bar{E}\gamma\beta\mu(\dot{q}_a)W_\beta + A + A^\perp \\
&\leq -\left[4\gamma\beta\bar{E}\mu(\dot{q}_a) - \frac{2}{\beta} \left(\frac{\|F^\perp(t)\|_a}{\sqrt{2\bar{E}}} - \min(0, \delta(t)) \right) \right] \\
&\quad \times W_\beta + \langle F^\perp, e_\beta \rangle \quad (55)
\end{aligned}$$

as required. \blacksquare

Proof: (Theorem 4) Consider the Lyapunov function $W_\beta = \langle e_\beta, e_\beta \rangle_a / 2$. Without loss of generality, assume that γ and β are nonnegative.

Using Lemma 2 and the bounds on F_e^α

$$\begin{aligned}
\frac{d}{dt} W_\beta &\leq -2\beta(2\bar{E}\gamma\mu(\dot{q}_a) - a - c_1)W_\beta \\
&\quad + \sqrt{2}\|F^\perp(t)\|_a W_\beta^{1/2} \quad (56)
\end{aligned}$$

where $W_\beta^{1/2}$ denotes the positive square root of W_β . We will first guarantee that $\mu(\dot{q}_a)$ is bounded from 0. Define

$$G(t) := 1 - \mu(\dot{q}_a(t)) = \frac{W_\beta(t)}{4\beta^2(t)\bar{E}}$$

so that $\mu(\dot{q}_a(t)) = 1 - G(t)$. It is readily shown using (55) and the bounds on the dissipation effect of F_e^α (Assumption ii) and on $\|F^\perp\|_a$ [Assumption iii]) that

$$\begin{aligned}
\frac{d}{dt} G(t) &\leq -2\beta(2\bar{E}\gamma(1 - G(t)) - a - c_1 - b)G(t) \\
&\quad + \beta c_1 G^{\frac{1}{2}}(t).
\end{aligned}$$

Choose any $\lambda > 0$, $G_0 \in (0, 1)$, and

$$\gamma_1 = \frac{1}{2\bar{E}(1 - G_0)} \left(\frac{c_1}{2G_0^{1/2}} + a + b + c_1 + \lambda \right).$$

Thus, if $\gamma > \gamma_1$, and $G(t) \leq G_0$

$$\frac{d}{dt} G(t) \leq -2\beta \left(\lambda G + \frac{c_1}{2} \left(\frac{G^{1/2}}{G_0^{1/2}} - 1 \right) G^{1/2} \right).$$

Notice that the right-hand side is nonpositive whenever $G(t) \leq G_0$. Therefore we can conclude that if $G(0) \leq G_0$, then $G(t) \leq G_0$ for all $t \geq 0$. This in turn, shows that $\mu(\dot{q}_a(t)) \geq (1 - G_0) > 0$. Having ensured that $\mu(\dot{q}_a)$ is positive, we can now

choose lower bound for γ to achieve the desired convergence and boundedness properties.

If $\gamma > \gamma_1$, $G(0) \leq G_0$ and $\|F^\perp\|_a \leq c_2\beta\sqrt{2\bar{E}}$

$$\frac{d}{dt} W_\beta \leq -2\beta \left[(2\bar{E}\gamma(1 - G_0) - a - c_1) W_\beta - c_2\sqrt{\bar{E}}W_\beta^{\frac{1}{2}} \right].$$

Consider first the case when $F^\perp = 0$, i.e., $c_1 = c_2 = 0$. Here, if $\gamma > \gamma_1$, and $G(0) \leq G_0$, then we have

$$\frac{d}{dt} W_\beta \leq -2\beta\lambda W_\beta.$$

Thus, W_β is monotonically decreasing and must converge. Moreover, we have

$$0 \leq W_\beta(t) \leq \exp\left(-\int_0^t \beta(\tau)\lambda d\tau\right) W_\beta(0).$$

Therefore, if $\int_0^t \beta(\tau) d\tau \rightarrow \infty$, $e_\beta(t) \rightarrow 0$. Otherwise, since $(d/dt)\beta^2(t)$ is bounded, if $\int_0^t \beta(\tau) d\tau \not\rightarrow \infty$, we must have $\beta(t) \rightarrow 0$. This also implies that $e_\beta(t) \rightarrow 0$.

When c_1, c_2 are not necessarily 0, given $\epsilon > 0$, the desired bound for e_β , choose $\lambda > 0$ and

$$\bar{\gamma} = \frac{1}{2\bar{E}(1 - G_0)} \left(\frac{c_2\sqrt{2\bar{E}}}{\epsilon} + a + c_1 + \lambda \right).$$

Then if $\gamma > \gamma_2 := \max(\gamma_1, \bar{\gamma})$,

$$\frac{d}{dt} W_\beta \leq -2\beta \left\{ \left[\lambda + \frac{c_2\sqrt{2\bar{E}}}{\epsilon} \right] W_\beta - c_2\sqrt{\bar{E}}W_\beta^{1/2} \right\}$$

where the right-hand side ≤ 0 whenever $W_\beta \geq \epsilon^2/2$ or $\|e_\beta\|_a \geq \epsilon$.

This implies that if $G(0) = W_\beta(0)/\beta^2(0) \leq G_0$ and $e_{\beta(0)}(0) \leq \epsilon$, $e_{\beta(t)}(t) \leq \epsilon$ for all $t \geq 0$.

On the other hand, if $G(0) \leq G_0$, whenever $\|e_\beta(t)\|_a > \epsilon$

$$\frac{d}{dt} W_\beta \leq -\beta\lambda\epsilon^2.$$

Hence, if in addition, $\beta(t) \geq \underline{\beta} > 0, \forall t$, this gives an estimate for a $T \geq 0$ such that for all $t > T$, $\|e_\beta(t)\|_a \leq \epsilon$. The estimate is $T = (e_\beta(0)^2 - \epsilon^2)/(2\underline{\beta}\lambda\epsilon^2)$. \blacksquare

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