OPTIMAL H_∞ CONTROL FOR LINEAR PERIODICALLY TIME-VARYING SYSTEMS IN HARD DISK DRIVES

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ABSTRACT

This paper discusses optimal H_∞ control synthesis via discrete Riccati equations for discrete linear periodically time-varying (LPTV) systems. Based on the results presented in [1], an explicit minimum entropy H_∞ controller for general time-varying systems is obtained. The control synthesis technique is subsequently applied to LPTV systems and it is shown that the resulting controllers are also periodically time varying. In order to demonstrate the effectiveness of the proposed control synthesis technique, both single-rate and multi-rate discrete-time minimum entropy H_∞ track-following control designs for hard disk drives are considered. It is shown, via a comprehensive simulation study, that track-following controllers designed using the H_∞ synthesis technique proposed in this paper achieve the robust performance of a desired error rejection function. Moreover, as expected, multi-rate controllers has the ability of outperforming their single-rate counterparts.

1 INTRODUCTION

This paper considers the optimal H_∞ control design of discrete linear periodically time-varying systems. This class of systems is frequently encountered in mechatronic systems such as hard disk drives (HDDs), where the rotation of the disks induces periodic dynamic phenomena. Moreover, as illustrated in this paper, HDD servo systems with multi-rate sampling and actuation [6] can be conveniently represented as linear periodically time-varying systems for control synthesis purposes.

H_∞ control is a popular control design methodology for synthesizing control systems that achieve robust performance or stabilization. In addition, through the use of classical loop-shaping techniques, it is possible to design H_∞ controllers that attain robust performance, while simultaneously satisfying a desired minimum level of error rejection loop shaping. Such techniques are potentially attractive in the design of mass-market mechatronic devices, such as HDDs, where consistent performance must be attained among tens of thousands of units in a given product line [10]. Since the pioneering work of Zames in [3], significant progress has been made in the design of optimal H_∞ control. In [4], a state-space solution to standard H_∞ control problems was given for continuous linear time-invariant (LTI) systems. As stated in [5], even though H_∞ control problems for discrete-time LTI systems can be solved by using the well-known bilinear transformation, it is more beneficial to solve the problems directly in the discrete-time domain. Peters and Iglesias [1] considered H_∞ control synthesis techniques for discrete time linear time-varying (LTV) systems via the minimum-entropy control paradigm. The authors provided a procedure for solving output feedback H_∞ control synthesis problems for discrete LTV systems, by breaking the overall problem into a series of simpler control problems, whose solutions were derived and known. These involve the output estimation control problem, the disturbance feedforward control problem, and the full information control problem. This paper utilizes the results and ideas presented in [1] to derive explicit and implementable solutions for the minimum entropy H_∞ control of linear periodic time varying

(LPTV) systems, using Riccati equations. Although it is possible to solve these types of problems as semi-definite programs (SDP) involving linear matrix inequality (LMI) constraints [7], we have found that the solutions via Riccati equations are often more computationally efficient and accurate than their corresponding SDP solutions [8].

The minimum entropy H_∞ control synthesis techniques developed for LPTV systems in this paper are used to design optimal H_∞ track-following controllers for both single-rate and multi-rate HDD servo systems [9], in order to evaluate their effectiveness. Although not discussed in this paper, the control synthesis techniques presented in this paper may also be useful for designing robust HDD servo systems with irregular sampling rates, due to missing position error signal samples [11].

This paper is organized as follows: Section 2 provides preliminary background that is necessary for developing the rest of the results contained in the paper. In Section 3, the minimum entropy optimal H_∞ control for LPTV systems is explicitly determined. The control methodology presented in Section 3 is used to design H_∞ track-following controller for HDDs in Section 4. Conclusions are given in Section 5.

2 PRELIMINARIES

Throughout this paper we assume that all input disturbances belong to ℓ_2, the set of all square summable sequences, and will consider discrete linear periodically time-varying systems that admit a state space realization with periodically time-varying entries as shown in (1),

$$G \sim \begin{bmatrix} x(k+1) \\ z(k) \\ y(k) \end{bmatrix} = \left[\begin{array}{c|cc} A(k) & B_1(k) & B_2(k) \\ \hline C_1(k) & D_{11}(k) & D_{12}(k) \\ C_2(k) & D_{21}(k) & 0 \end{array}\right] \begin{bmatrix} x(k) \\ w(k) \\ u(k) \end{bmatrix} \tag{1}$$

where $w(k)$ and $u(k)$ are respectively the disturbance and control inputs, $y(k)$ is the measurable output, which is accessible to the control system and $z(k)$ is "performance monitoring" output, used in our optimization cost function. All time-varying entries in G are assumed to be periodic with period N, for example, $A(k) = A(k+N)$. As in [1], bold operators will denote the corresponding block diagonal representations of time varying matrices, for example, $\boldsymbol{A} = \mathrm{diag}\{A(k)\}_{k=0}^{\infty}$. The operator $\boldsymbol{A}$ is uniformly exponentially stable (UES) if there exist constants $c>0$ and $\beta \in [0,1)$ such that for all nonnegative integers $l, \beta \in Z^+$.

$$\| A(k+l-1)A(k+l-2)\cdots A(k) \| \le c\beta^l$$

The pair $(\boldsymbol{A}, \boldsymbol{B})$ is uniformly stabilizable, if there exists a bounded memoryless operator $\boldsymbol{F}$ such that $\boldsymbol{A} + \boldsymbol{BF}$ is UES. Likewise, the pair $(\boldsymbol{A}, \boldsymbol{C})$ is uniformly detectable, if there exists a bounded memoryless operator $\boldsymbol{H}$ such that $\boldsymbol{A} + \boldsymbol{HC}$ is UES.

For discrete time LPTV systems, the H_∞ norm is generalized as the ℓ_2 induced norm [1]. For an LPTV operator $\boldsymbol{H}$ with input $w(k)$ and output $z(k)$, its ℓ_2 induced norm is defined as

$$\|H\|_{2\leftarrow 2} = \left(\sup_{w \in \ell_2 \setminus \{0\}} \frac{\sum_{k=0}^{\infty} z^T(k) z(k)}{\sum_{k=0}^{\infty} w^T(k) w(k)} \right)^{1/2} \tag{2}$$

Given the LPTV plant defined in (1), the control objective is to find an optimal linear time-varying compensator $\boldsymbol{K}$ with input $y(k)$ and output $u(k)$ so that the ℓ_2 induced norm of the closed loop system $\mathcal{F}_\ell(G,K)$, from the disturbance input $w(k)$ to the output $z(k)$ as depicted in Fig. 1, is less than $\gamma>0$, i.e.

$$\min_{K,\gamma} \gamma$$
$$\text{s.t. } \|\mathcal{F}_\ell(G,K)\|_{2\leftarrow 2} < \gamma . \tag{3}$$

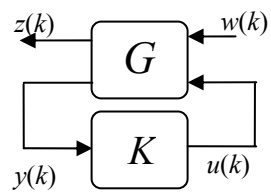

FIGURE 1. CONTROL DESIGN FORMULATION

A key element of the minimum entropy H_∞ control synthesis design methodology presented in [1], is to verify the existence of a controller that satisfies the inequality constraint in (3) for a given $\gamma>0$. After developing a criterion for checking the existence of such controllers, we can utilize bisection search algorithms to obtain the minimum γ^* (t^* represents the corresponding optimal value of t). In order to simplify our algorithm for control synthesis, we will assume that $\gamma = 1$ in (3). If not, we can perform the scaling $\gamma^{-1/2}B_1$, $\gamma^{1/2}B_2$, $\gamma^{-1/2}C_1$, $\gamma^{1/2}C_2$, $\gamma^{-1}D_{11}$, and then obtain the compensator $\gamma^{-1}K$.

Throughout this paper, we will use the following notation, $B(k) = \begin{bmatrix} B_1(k) & B_2(k) \end{bmatrix}$, $D_{1\bullet}(k) = \begin{bmatrix} D_{11}(k) & D_{12}(k) \end{bmatrix}$, $C(k) = \begin{bmatrix} C_1(k) \\ C_2(k) \end{bmatrix}$, and $D_{\bullet 1}(k) = \begin{bmatrix} D_{11}(k) \\ D_{21}(k) \end{bmatrix}$.

In addition, we will use the following fairly standard assumptions from [1], regarding the H_∞ control design of the time-varying system in (1):

A1. $D_{12}^T(k)D_{12}(k) \succ 0$ and $D_{21}(k)D_{21}^T(k) \succ 0$ for all k,

A2. the pair $(\boldsymbol{A}, \boldsymbol{B}_2)$ is uniformly stabilizable and the pair $(\boldsymbol{C}_2, \boldsymbol{A})$ is uniformly detectable,

A3. the pair $\left(\boldsymbol{A}-\boldsymbol{B}_1\boldsymbol{D}_{21}^{\dagger}\boldsymbol{C}_2,\quad \boldsymbol{B}_1\left(\boldsymbol{I}-\boldsymbol{D}_{21}^{\dagger}\boldsymbol{D}_{21}\right)\right)$ is uniformly stabilizable with $\boldsymbol{D}_{21}^{\dagger}=\boldsymbol{D}_{21}^{T}\left(\boldsymbol{D}_{21}\boldsymbol{D}_{21}^{T}\right)^{-1}$,

A4. the pair $\left(\left(\boldsymbol{I}-\boldsymbol{D}_{12}\boldsymbol{D}_{12}^{\dagger}\right)\boldsymbol{C}_1,\quad \boldsymbol{A}-\boldsymbol{B}_2\boldsymbol{D}_{12}^{\dagger}\boldsymbol{C}_1\right)$ is uniformly detectable with $\boldsymbol{D}_{12}^{\dagger}=\left(\boldsymbol{D}_{12}^{T}\boldsymbol{D}_{12}\right)^{-1}\boldsymbol{D}_{12}^{T}$.

3 OPTIMAL H∞ CONTROL FOR DISCRETE TIME LPTV SYSTEMS

In order to use the minimum entropy control synthesis techniques for general time-varying systems presented in [1], we will temporarily ignore the periodicity of the LPTV system in (1). As stated in [1], the output feedback H_∞ control for general discrete LTV systems is synthesized in three steps: 1) the output feedback control problem is transformed to an output estimation control problem; 2) the solution to the output estimation control problem is obtained as the dual of the disturbance feedforward control problem; 3) the disturbance feedforward control problem is obtained by solving the full information control problem, whose solution was obtained in [1]. However, no explicit formulae are presented in [1] for controllers synthesized using these steps. Alternatively, the solution to the output feedback control problem can be obtained by transforming the output feedback control problem to an output estimation control problem, and then the solution to the output estimation control problem can be directly obtained from the solution to the full control problem, which is easily obtained from the results in [1]. Utilizing the latter procedure, yielded the following unique stabilizing minimum entropy time-varying controller K, which satisfies the constraint in (3) and is given by the following state space realization:

$$\begin{cases}\hat{x}(k+1)=\bar{A}(k)\hat{x}(k)+B_2(k)u(k)+F_t(k)\left(\bar{C}_2(k)\hat{x}(k)-y(k)\right)\\ u(k)=-T_{22}^{-1}(k)\bar{C}_{12}(k)\hat{x}(k)+L_t(k)\left(\bar{C}_2(k)\hat{x}(k)-y(k)\right)\end{cases} \quad (4)$$

The parameters used to construct the controller are updated in the following steps:

S1. Solve backwards in time and store the state feedback Riccati equation solution for all j:

$$X(j)=A^T(j)X(j+1)A(j)+C_1^T(j)C_1(j)$$
$$-M(j)\left(R(j)+B^T(j)X(j+1)B(j)\right)^{-1}M^T(j)$$

where $M(j)=A^T(j)X(j+1)B(j)+C_1^T(j)D_{1\bullet}(j)$ and $X(j)\geq 0$.

After obtaining the solution $X(j)$ for all $j=0,1,2,\ldots$, we continue to calculate the other parameters.

S2. Define $T(k)=\begin{bmatrix}T_{11}(k) & 0\\ T_{21}(k) & T_{22}(k)\end{bmatrix}, T_{11}(k)\succ 0, T_{22}(k)\succ 0$.

S3. Compute $T(k)$ using:

$$R(k)+B^T(k)X(k+1)B(k)=T^T(k)J(k)T(k),$$

where $R(k)=D_{1\bullet}^T(k)D_{1\bullet}(k)-\begin{bmatrix}I & 0\\ 0 & 0\end{bmatrix}, J(k)=\begin{bmatrix}-I & 0\\ 0 & I\end{bmatrix}$.

S4. Get $\begin{bmatrix}F_1(k)\\ F_2(k)\end{bmatrix}=-\left(R(k)+B^T(k)X(k+1)B(k)\right)^{-1}M^T(k)$.

S5. Calculate the following matrices for the filtering Riccati equation: $\bar{A}(k)=A(k)+B_1(k)F_1(k)$, $\bar{C}_2(k)=C_2(k)+D_{21}(k)F_1(k)$ and $\bar{C}_{12}(k)=-T_{22}(k)F_2(k)$. Let $D_\perp(k)$ be an orthogonal matrix to $D_{12}(k)$. In addition, define a matrix $W(k)$ such that $W^T(k)W(k)=I-T_{11}^T(k)T_{11}(k)$ and $W(k)$ has appropriate dimensions so that the following matrix multiplication is well defined $\begin{bmatrix}\bar{D}_{111}(k)\\ \bar{D}_{112}(k)\end{bmatrix}=D_\perp(k)W(k)+D_{12}(k)T_{21}(k)$.

S6. Update forwards in time the filtering Riccati equation solution:

$$Y(k)=\bar{A}(k)Y(k-1)\bar{A}^T(k)+B_1(k)B_1^T(k)$$
$$-\tilde{M}(k)\left(\tilde{R}(k)+\begin{bmatrix}\bar{C}_{12}(k)\\ \bar{C}_2(k)\end{bmatrix}Y(k-1)\begin{bmatrix}\bar{C}_{12}(k)\\ \bar{C}_2(k)\end{bmatrix}^T\right)^{-1}\tilde{M}^T(k),$$

where

$$\tilde{M}(k)=\bar{A}(k)Y(k-1)\begin{bmatrix}\bar{C}_{12}(k)\\ \bar{C}_2(k)\end{bmatrix}^T+B_1(k)\begin{bmatrix}\bar{D}_{112}(k)\\ D_{21}(k)\end{bmatrix}^T \text{ and}$$

$Y(k)\geq 0$.

S7. Define $\tilde{T}(k)=\begin{bmatrix}\tilde{T}_{11}(k) & \tilde{T}_{12}(k)\\ 0 & \tilde{T}_{22}(k)\end{bmatrix}, \tilde{T}_{11}(k)\succ 0, \tilde{T}_{22}(k)\succ 0$.

S8. Compute $\tilde{T}(k)$ using:

$$\tilde{R}(k)+\begin{bmatrix}\bar{C}_{12}(k)\\ \bar{C}_2(k)\end{bmatrix}Y(k-1)\begin{bmatrix}\bar{C}_{12}(k)\\ \bar{C}_2(k)\end{bmatrix}^T=\tilde{T}(k)\tilde{J}(k)\tilde{T}^T(k),$$

where

$$\tilde{R}(k)=\begin{bmatrix}\bar{D}_{112}(k)\\ D_{21}(k)\end{bmatrix}\begin{bmatrix}\bar{D}_{112}(k)\\ D_{21}(k)\end{bmatrix}^T-\begin{bmatrix}I & 0\\ 0 & 0\end{bmatrix}, \tilde{J}(k)=\begin{bmatrix}-I & 0\\ 0 & I\end{bmatrix}.$$

S9. Obtain $\begin{bmatrix}\tilde{F}_1(k)\\ \tilde{F}_2(k)\end{bmatrix}=$

$$-\left(\tilde{R}(k)+\begin{bmatrix}\bar{C}_{12}(k)\\ \bar{C}_2(k)\end{bmatrix}Y(k-1)\begin{bmatrix}\bar{C}_{12}(k)\\ \bar{C}_2(k)\end{bmatrix}^T\right)^{-1}\tilde{M}^T(k).$$

S10. Calculate the filter gains $L_t(k)=T_{22}^{-1}(k)\tilde{T}_{12}(k)\tilde{T}_{22}^{-1}(k)$ and $F_t(k)=\tilde{F}_1^T(k)\tilde{T}_{12}(k)\tilde{T}_{22}^{-1}(k)+\tilde{F}_2^T(k)$.

Details of the controller derivation are omitted due to space constraints.

It should be noted that, because we are solving an infinite horizon problem and the state feedback Riccati equation given in

S1 must be solved backwards in time, its exact solution does not currently exist [1] and thus the controller in (4) is not implementable for general time-varying systems. However, for LPTV systems, the stabilizing solutions to the Riccati equations in S1 and S6 are unique [1]. Moreover, as shown in Lemma 1, these solutions are also periodic. As a result, the solutions to two Riccati equations converge to the corresponding stabilizing solutions, which can be solved in a straight forward manner by iteration, starting respectively from arbitrary symmetric and positive semidefinite final and initial conditions.

Lemma 1 For LPTV systems with period N, the H_∞ controller given by (4) is also periodic with period N.

Proof: Firstly, we will show the periodicity of the solution to the discrete Riccati equation in S1. Suppose $(\cdots, X(k), X(k+1), \cdots)$ is the solution to the discrete Riccati equation in S1, which means that

$$X(k) = A^T(k)X(k+1)A(k) + C_1^T(k)C_1(k) - \eta_k\left(X(k+1)\right),$$

where

$\eta_k\left(X(k+1)\right) = M(k)\left(R(k) + B^T(k)X(k+1)B(k)\right)^{-1} M^T(k)$. At the time of $k+N$, we have:

$$X(k+N) = A^T(k+N)X(k+N+1)A(k+N) + C_1^T(k+N)C_1(k+N) - \eta_{k+N}\left(X(k+N+1)\right).$$

By considering that the plant G is periodic with period N,

$$X(k+N) = A^T(k)X(k+N+1)A(k) + C_1^T(k)C_1(k) - \eta_k\left(X(k+N+1)\right)$$

which implies $(\cdots, X(k+N), X(k+N+1), \cdots)$ is another solution to S1. From [1], we know that the bounded stabilizing solution to S1 is unique, which implies $X(k) = X(k+N)$.

Therefore, all $\bar{A}(k), \begin{bmatrix} \bar{C}_{12}(k) \\ \bar{C}_2(k) \end{bmatrix}$, and $\begin{bmatrix} \bar{D}_{112}(k) \\ D_{21}(k) \end{bmatrix}$ in S5 are periodic with period N. The periodicity of the solution to the discrete Riccati equation in S6 can be shown in a similar manner, i.e. $Y(k) = Y(k+N)$. As a result, the periodicity of $X(k)$ and $Y(k)$ implies that all of the parameters to construct the H_∞ controller in (4) are periodic with period N. Therefore, the H_∞ controller for the LPTV system G is also periodic with period N.

As expected, the periodicity of H_∞ controllers for LPTV systems provides a significant advantage since the Riccati equations in S1 and S6 can be solved backward and forward respectively with zero initial conditions by iteration, and their solutions will converge to the corresponding periodic solutions.

4 DESIGN EXAMPLES FOR HDD SERVO SYSTEMS

The plant dynamics of voice coil motors (VCM) and actuators in hard disk drives have multiple structural resonance modes, which are subjected to unit-to-unit varying natural frequencies and damping ratios, due to manufacturing tolerances in mass production and changing operating conditions. Therefore, the control design of HDD servo systems must take performance robustness into account. As will be illustrated subsequently, optimal H_∞ control is quite suitable for HDD servo systems, since it is a convenient methodology for performing loop-shaping design and achieving a desired error rejection transfer function for a set of HDDs with plant variations.

In order to evaluate our proposed optimal H_∞ control design methodology in this paper, the algorithm will be tested via a simulation study that utilizes a HDD benchmark model developed by the IEEJapan technical committee on Nano-Scale Servo (NSS) system [12]. The nominal VCM model is indicated in Fig. 2. In the simulations, we will assume that the position error signal (PES) sampling frequency is f_s = 26400 Hz.

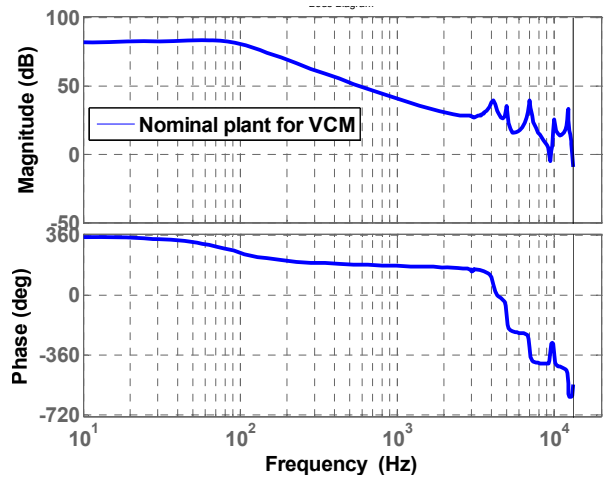

FIGURE 2. NOMINAL VCM MODEL

4.1 Optimal H_∞ track-following control for single-rate HDD servos

We consider the block diagram shown in Fig. 3 for the optimal H_∞ control design, where G_v^n, W_p, W_u, and W_Δ are respectively the nominal VCM plant, loop-shaping performance weighting function, control input weighting value, and plant uncertainty weighting function. In this example we consider the case when the control actuation is performed at the same rate as the PES sampling rate.

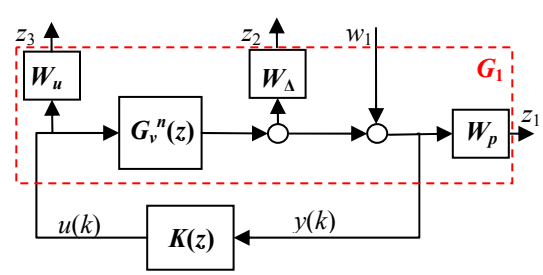

FIGURE 3. CONTROL DESIGN FORMULATION FOR SINGLE-RATE HDDS

Thus, the single-rate servo system can be interpreted as an LPTV system with period $N = 1$. The LPTV system with period 1 turns out to be an LTI system. As a result, the corresponding control problem can be stated as:

$$\min_{\gamma, K} \gamma$$
$$\text{s.t.} \quad \left\| T_{z \leftarrow w_1} \right\|_\infty < \gamma$$

$T_{z \leftarrow w_1}$ represents the transfer function matrix from w_1 to $z = \begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix}^T$ as shown in Fig. 3.

Notice that with $N = 1$, all of entries in the state-space realization of (1) are constant, and thus the Riccati equation solutions in S1 and S6 converge to steady state solutions that can be computed via their corresponding discrete algebraic Riccati equations (DAREs). Consequentially, the synthesized optimal H_∞ controllers for LTI systems are also time-invariant.

Assuming that the weighting functions W_p and W_Δ and the weight W_u, are chosen so that the H_∞ synthesis technique yields a solution to the minimization problem in (3) with $\gamma \leq 1$. Then, we obtain

$$\begin{cases} \left\| (1 + G_v{}^n K)^{-1} \right\|_\infty \cdot \left\| W_p \right\|_\infty < 1 \\ \left\| G_v{}^n K (1 + G_v{}^n K)^{-1} \right\|_\infty \cdot \left\| W_\Delta \right\|_\infty < 1 \end{cases} \tag{5}$$

These equations are the basis for performing loop-shaping design and achieving a desired error rejection transfer function (also known as the sensitivity transfer function). Notice that, for SISO systems, $|W_\Delta(e^{j\omega})|$ should be larger than the plant output multiplicative uncertainty magnitude, while ideally, $|W_p(e^{j\omega})|^{-1}$ should be larger than the magnitude of the desired error rejection transfer function.

The magnitude Bode plot of the performance weighting function inverse for the single-rate design is indicated by the blue line in Fig. 4. Based on (5), we have selected W_p so that the magnitude Bode plot of the designed error rejection transfer function will be below that of $|W_p(\omega)|^{-1}$ and its crossover frequency will be at least 1.3 KHz. The magnitude Bode plot of the uncertainty weighting function W_Δ considered in this control design example is shown in Fig. 5, which indicates that the plant could have unstructured output uncertainty variation as high as ±8% at low frequency and ±12% at high frequency. The weighting value W_u is to be tuned so that the achieved γ is less than or equal to 1 and simultaneously the control actuation generated by the resulting control K is appropriate under the hardware constraints of real HDD servo systems.

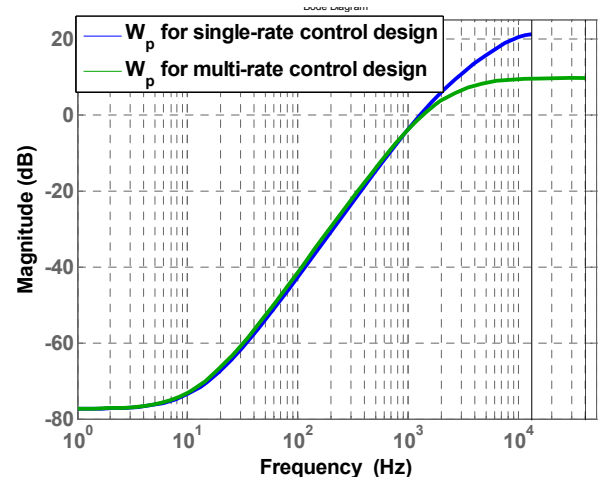

FIGURE 4. PERFORMANCE WEIGHTING FUNCTIONS

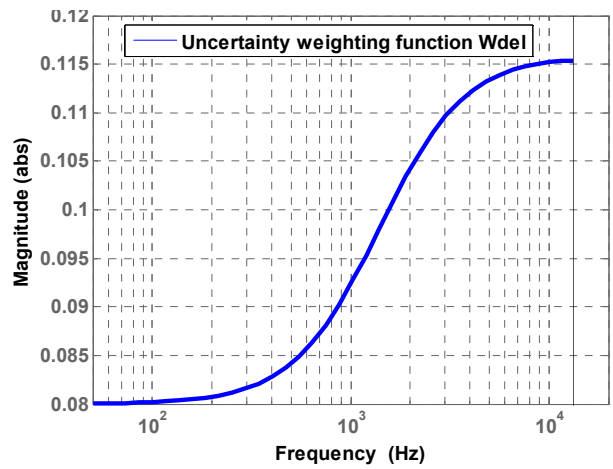

FIGURE 5. UNCERTAINTY WEIGHTING FUNCTION

In order to investigate the accuracy of the control synthesis algorithm presented in this paper, we will compare a controller that is synthesized by our proposed methodology to the one that is synthesized by the Matlab function of "hinfsyn" in Robust Control Toolbox using identical plant parameters and weighting functions. Notice that "hinfsyn" function in the Matlab version of R2007b performs H_∞ control synthesis for discrete time systems by first mapping the discrete time plant to a continuous time plant using the bilinear transformation, performing H_∞ control synthesis in continuous time and then mapping the resulting H_∞ continuous time control back to discrete time using the bilinear transformation.

Figure 6 shows the magnitude Bode plot of the closed-loop error rejection transfer functions when the H_∞ controller is synthesized using the methodology proposed in this paper and when it is synthesized using the Matlab function "hinfsyn." As shown in the figure, the Matlab function "hinfsyn" failed to synthesize a controller that could satisfy all constraints in (5) (as shown by the green line in Fig. 6), while the synthesis technique

proposed in this paper produced a minimum entropy H_∞ controller that satisfied all constraints with an optimal γ^*=1.0 (as shown by the blue line in Fig. 6). As stated in [5], the successive uses of the bilinear transformation (discrete time to continuous time and then back to discrete time) in the Matlab function "hinfsyn" may have introduced numeric accuracy problems, particularly when the systems are ill-conditioned. We have found in our simulation study that when the H_∞ constraints are relaxed, for example by selecting "less aggressive" performance weighting function W_p, the H_∞ controllers produced by the two synthesis techniques are almost identical.

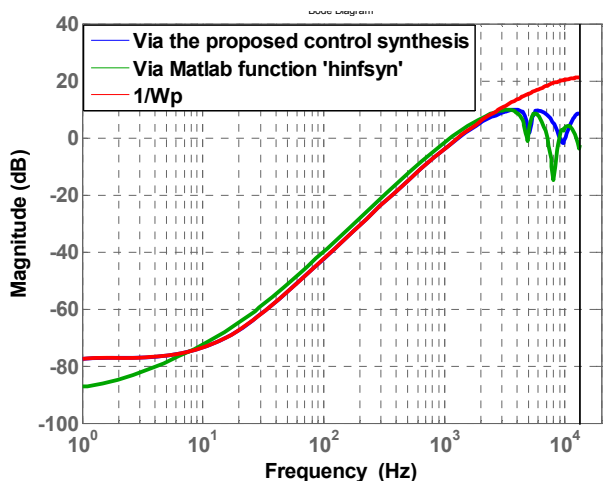

FIGURE 6. SENSITIVITY FUNCTIONS FOR SINGLE-RATE HDD SERVO SYSTEMS

4.2 Optimal H_∞ track-following control for multi-rate HDD servos

As discussed in [13], increasing the control actuation rate can improve both track-following control performance and robustness. In this section we test this idea by comparing the single-rate H_∞ controller that was designed in the previous section with a multi-rate H_∞ controller, where the actuation rate f_a is three times higher than the PES sampling rate f_s. In addition, since the higher rate actuation could improve the control performance, a little more aggressive performance weighting function indicated by the green line in Fig. 4 has been utilized for the multi-rate control design. Because of the higher actuation rate, it is necessary to discretize the plant model and the weighting functions, indicated in the block diagram in Fig. 3, at the higher actuation rate f_a and to assume that PES measurements are only available at the instances $k \in \{\ldots, -N, 0, N, 2N\ldots\}$ where N=3. Then the open loop plant, indicated by the dashed box in Fig. 3, for this servo system can be represented by following LPTV system with period $N = 3$.

$$G_1 \sim \begin{bmatrix} x(k+1) \\ z(k) \\ y(k) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1^o & B_2 \\ \hline C_1 & D_{11}^o & D_{12} \\ C_2^o(k) & D_{21}^o(k) & 0 \end{array}\right] \begin{bmatrix} x(k) \\ w_1(k) \\ u(k) \end{bmatrix} \tag{6}$$

Notice that all matrices in (6) are constants except for $C_2^o(k)$ and $D_{21}^o(k)$, where

$$\begin{bmatrix} C_2^o(k) & D_{21}^o(k) \end{bmatrix} = \begin{cases} \begin{bmatrix} C_{2m} & D_{21m} \end{bmatrix}, k \in \{\ldots, -N, 0, N, 2N\ldots\} \\ \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad \text{otherwise} \end{cases} \tag{7}$$

From the standard assumption in A1, the derived controller requires a condition $D_{21}(k)D_{21}^T(k) \succ 0$ for all k. Unfortunately, the typical approach to enforcing multi-rate sampling and actuation [14] will not work here because this would result in $D_{21}(k)D_{21}^T(k)$ being singular. In order to deal with this constraint, we introduce a fictitious disturbance input w_2 which will be injected into the feedback signal y when the PES is not available, as shown in Fig. 7. Notice that this fictitious noise will not affect the minimum closed-loop induced norm and the resulting optimal H_∞ controller, if the gain of the resulting time varying controller $K_k(z)$ is zero when the PES measurement is unavailable at the time of k. This requirement has been achieved when the minimum entropy H_∞ control synthesis proposed in this paper is used, as will be discussed in detail later.

Unlike the standard H_∞ control problem formulation [2], in this paper the performance weighting function is moved to the disturbance input side. By doing so, the feedback signal y is affected by a "colored" disturbance ($W_p w_1$) more directly when the sampling rate is lower than the actuation rate. In order to maintain the same H_∞ constraints as those that were used in Section 4.1, it is necessary to use the control synthesis architecture represented in Fig. 7 with the fictitious disturbance w_2, where the weighting functions W_p, W_u, and W_Δ are the same as the weighting functions in Section 4.1.

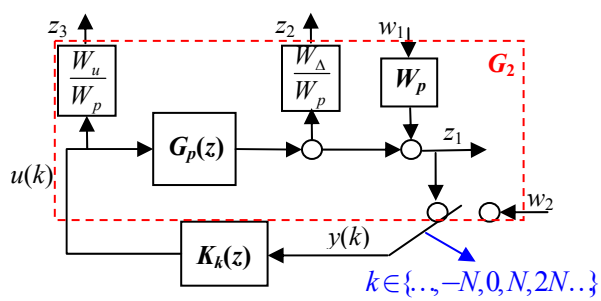

FIGURE 7. CONTROL DESIGN FORMULATION FOR HDDS WITH MULTI-RATE SAMPLING AND ACTUATION

As a result, we can formulate the optimal H_∞ control design problem as shown in (3) by replacing the system G by G_2, where G_2 is indicated by the dashed box in Fig. 7. Thus G_2 is the open loop map from $w = \begin{bmatrix} w_1 & w_2 \end{bmatrix}^T$ to $z = \begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix}^T$, which can be represented by the following LPTV system with period $N = 3$.

$$G_2 \sim \begin{bmatrix} x(k+1) \\ z(k) \\ y(k) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2(k) & D_{21}(k) & 0 \end{array}\right] \begin{bmatrix} x(k) \\ w_1(k) \\ u(k) \end{bmatrix} \tag{8}$$

The corresponding matrices in (8) for the measurement changes to be:

$$\begin{bmatrix} C_2(k) & D_{21}(k) \end{bmatrix} = \begin{cases} \begin{bmatrix} C_{2m} & \begin{bmatrix} D_{21m} & 0 \end{bmatrix} \end{bmatrix}, k \in \{k : PES \text{ is available}\} \\ \begin{bmatrix} 0 & \begin{bmatrix} 0 & 1 \end{bmatrix} \end{bmatrix}, \qquad \text{otherwise} \end{cases} \quad (9)$$

Notice that all matrices in (8) are constants except for $C_2(k)$ and $D_{21}(k)$. Therefore, since all the matrices involved in the computation of the state feedback Riccati equation solution in S1 are constant, the solution $X(k)$ will converge to a constant matrix which can be computed via its corresponding DARE. Consequentially, the parameters X, T and F that are respectively computed in S1-S4 are constant, which implies the parameters $\bar{A}$ and $T_{22}^{-1}\bar{C}_{12}$ in the proposed controller (4) will also be constant. In addition, the filtering Riccati equation solution $Y(k)$, which is computed forwards in time using S6 and a zero initial condition, will converge to a steady-state periodic solution with period N, as demonstrated by Lemma 1. It turns out in this case that the filter gains $F_t(k)$ and $L_t(k)$ will also be periodic with period N. Moreover, with (9) and the control structure given by (4), it is straightforward to show that $\tilde{T}_{12}(k)$ and $\tilde{F}_{12}(k)$ are both equal to 0 at the instances when the PES is not available. Thus, $F_t(k)$ and $L_t(k)$ will also be equal to 0 at these instances, justifying the use of the fictitious disturbance w_2 in the control synthesis methodology. As a result, the minimum closed-loop ℓ_2 induced norm and the optimal H_∞ controller are unaffected by the use of the fictitious disturbance w_2 in the control synthesis methodology.

Performing a multi-rate minimum entropy H_∞ control synthesis described above, produces a multi-rate H_∞ controller that returns an optimal ℓ_2 induced norm of γ^*=1.0. As an illustration, the values of the computed periodic time varying observer residual gain $L_t(k)$ of the controller given by (4) are shown in Fig. 8, indicating that $L_t(k)$ = 0 in the instances when PES is not available. Similar results were obtained for periodic time varying vector $F_t(k)$.

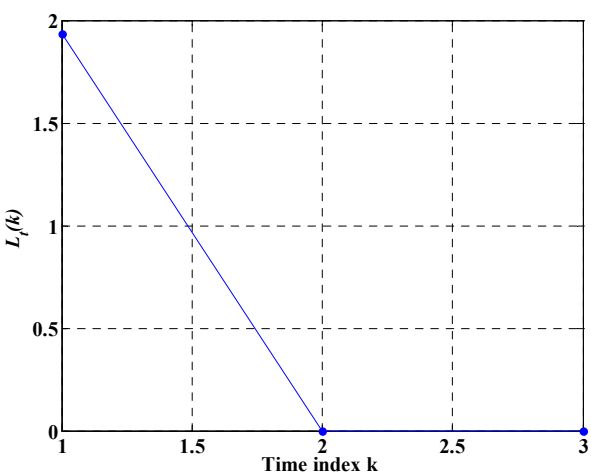

FIGURE 8. DESIGNED PERIODIC GAIN

With the more aggressive performance weighting function, the achieved optimal ℓ_2 induced norm of γ^* = 1.0 implies that the multi-rate strategy has the ability to improve the control performance. In order to furthermore highlight the performance and robustness improvements that can be attained by introducing a higher actuation rate, a time domain simulation study was performed using both the single rate minimum entropy H_∞ control design in Section 4.1 and the multi-rate minimum entropy H_∞ control design in this section. To evaluate the robust performance of the two control designs, each controller was tested on 50 different plants, which were randomly generated from the nominal plant shown in Fig. 2 with the output multiplicative uncertainty weighting function indicated in Fig. 5 by using Matlab function "usample". Note that the disturbance models for our time domain simulation are provided in [12].

Table 1 contains the root mean square (RMS) 3σ values of the PES for the nominal plant and the worst-case results for each controller. These results indicate that a controller with a higher actuation rate is able to not only reduce the 3σ PES by 12.5% for the nominal plant, but also improve the servo performance for the worst-case situation.

TABLE 1. SIMULATION RESULTS

	3σ PES (% of track)	
	Nominal	Worst
Single rate control design	10.4	10.8
Multi-rate control design with N=3	9.1	9.6

5 CONCLUSION

The optimal H_∞ control synthesis via discrete Riccati equations for discrete linear periodically time-varying systems was studied in this paper. Using the results in [1], an explicit minimum entropy H_∞ controller for general discrete linear time-varying systems was obtained. The control synthesis technique was subsequently applied to LPTV systems, and it was verified that the resulting controllers are also LPTV. The presented control synthesis technique was tested by designing both single-rate and multi-rate minimum entropy H_∞ track-following controllers for HDD servo systems. In the case of single-rate control, the H_∞ controller designed using the synthesis technique presented in this paper was compared to an H_∞ controller designed using the Matlab function of "hinfsyn." Simulation results showed that former synthesis technique is more numerically robust in calculating optimal discrete-time H_∞ controllers for discrete linear time-invariant systems than the latter. Moreover, the presented control synthesis algorithm is also applicable to multi-rate sampling and actuation, while the "hinfsyn" function in Matlab is only applicable for LTI plants. Simulation results demonstrate that multi-rate H_∞ controllers synthesized using the technique presented in this paper consistently has the ability of

outperforming their single-rate counterparts and offering improved robust performance.

ACKNOWLEDGMENTS

This work was performed with funding partially with support from the Information Storage Industry Consortium (INSIC) and UC Berkeley Computer Mechanics Laboratory (CML).

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