

An Equivalent Second Order Model Based on Flow and Speed

X. Y. Lu*, P. Varaiya**, and R. Horowitz***

*PATH, U. C., Berkeley, Richmond Field Station, Bldg 4521357 S. 46th Street, Richmond, CA 94804-4648, USA (Tel:510-665 3644; e-mail: xylu@path.berkeley.edu).

**EECS, U. C. Berkeley, 271M Cory Hall, Berkeley CA 94720-1770
USA (e-mail: variya@eecs.berkeley.edu)

***ME, U. C. Berkeley, 5138 Etcheverry Hall, Berkeley CA 94720-1770
USA (e-mail: horowitz@me.berkeley.edu)

Abstract: Traffic models developed for control include first order LWR model and second order models. The first order model is essentially density dynamics which states the conservation of the number of vehicles in a section of highway. Most second order models are essentially a speed dynamics coupled with the first order density dynamics. There are several second order models but most are for explaining traffic flow. Only a couple of second order models were developed for traffic control purpose. The most popular one is the Payne-Papageorgiou model. Due to point sensor detection limit in highways such as inductive loops, flow estimation is easy and reasonably accurate while density estimation is very difficult. This is the motivation for us to have a look at flow dynamics in place of density. It turns out that flow dynamics of first order cannot be well-defined contrasted to the Cell Transmission Model. However, a flow-speed dynamics can be obtained which is equivalent to the density-speed dynamics of the Payne-Papageorgiou model.

Keywords: traffic flow, density and speed; traffic flow model; first order Cell Transmission Model; second order Payne-Papageorgiou model,

1. INTRODUCTION

In recent years, model-based traffic control design has been becoming more and more popular. The analysis and control design of ramp metering based on the first order Cell Transmission Model (CTM) is one example (Muñoz et al, 2004; Gomes and Horowitz, 2006). Another example is to use a second order model for combined Variable Speed Limit and Coordinated Ramp Meter control design in (Papageorgiou, 1983, 1990), Papamichail et al (2008), and Hegyi (2005). This paper is to address some modelling issue based on practical traffic control design considerations.

The CTM is essentially a density dynamics. Thus corresponding control using ramp metering is to control the average density in each cell. For the control design using second order model, the density ρ and distance mean speed v are the state variables. By definition, density is a distance concept which, in principle, can be estimated by the vehicle count instantly captured by video camera for the stretch of highway involved. However, in practice, video camera is not easy to install and maintain. Most traffic sensors in the road are dominantly the inductive loops which provide point measurement. High density loop detector would give a better estimation of density but it is cost prohibitive. It is well-known that loop, particularly a dual loop station, has a good measurement of vehicle count in unit time, or flow. A question naturally arises: is it possible to use flow instead of density as state variable in traffic dynamics for control design? This paper provides an answer to this question. It

turns out that the flow cannot be the state in first order model, but can be coupled with speed to form a second order model.

The paper is organized as follows: section 2 is literature review; section 3 deduce the flow dynamics; section 4 discuss the possibility for flow to be a state in first order model, and coupled with speed dynamics to for a second order model; and section 5 is conclusions.

2. LITERATURE REVIEW

There are at least three traffic modelling approaches in the literature for macroscopic traffic modelling.

(1) Based on the physics of fluid flow – traffic is considered as compressible flow. Representative work in this approach is the well-known LWR model Lighthill and Whitham (1955a, b) and Richards (1956), and later development by Newell (1993a, b, c), Daganzo (1995a, b, c) and Zhang (1998, 1999a, 1999b, 2000). A good collection and review of the kinematic wave models and its development history can be found in Gartner et al (2001).

(2) Based on driver behaviour and intuitive understanding of traffic behaviour such as prediction and delay. Representative work in this direction is the model by Payne (1971) and the improvement by Papageorgiou (1983), which is called Payne-Papageorgiou model here. It is essentially a second model with coupled density and speed dynamics. Since then, several second order models have been obtained by some modification/improvement on the model. The model has been further improved in (Papageorgiou, 1990) by introducing the

waving effect due to lane changing. Since the authors intended for ramp meter control and different ramp meter strategy evaluation only, there was essentially only one control variable – the ramp meter rate. The speed dynamics is not intended to be independent. Instead, it generated a reference speed based a static density-speed relationship – the Fundament Diagram and went back to the loop to affect the density dynamics indirectly through some coupling. In this sense, the second model is essentially a dynamics of density with some driver behaviour added through the speed dynamics. The theoretical model of second order was used in (Hegyi et al, 2005) for combined VSL and Ramp Metering for reducing shockwave. A good reference is referred to (Nagel, 1998) for understanding several second order traffic model based on fluid dynamics. The explanation is very interesting, particularly the car following model and the Payne- Papageorgiou model.

(3) The third approach is to obtain second order model by aggregating microscopic traffic model such as car following model (Darbha, 2001; Tyagi et al 2008). Aggregation of a set of dynamical system must follow some rules: (a) their initial conditions must belong to the same set; (b) the dynamics should have the similar structure; and (c) Considering the trajectories and Math Flow, there exists a *monic* and an *epic* between two categories; for the collection of vehicle trajectories on freeways, the monic and epic exist. The second order model deduced and discretized in (Tyagi et al 2008) is different from any other second order model in the literature. Zhang (1998) considered the consistence of micro and traffic models for the improvement of second order model to remove the upstream traffic effect on downstream flow which is a gas like behaviour. It also falls into this category. The first order model is the conservation law; and the second dynamics on speed is a conservation of momentum:

$$v_t + vv_x = -c(\rho)v_x$$

which was claimed to be deduced from the following car following model:

$$\tau(s_n(t))\ddot{x}(t) = \dot{x}_{n-1}(t) - \dot{x}_n(t)$$

$$s_n(t) = x_{n-1}(t) - x_n(t)$$

$s_n(t)$ - following distance

$\tau(s_n(t))$ - driver response time

$-c(\rho) \equiv -\rho V'_*(\rho) = \frac{s}{\tau(s)}$ is an introduced concept – traffic

sound speed. However, the deduction simply dropped the index for individual vehicles, which is different from the principles suggested in (Darbha, 2001).

The most popularly used first order model for traffic control purpose is the CTM (Daganzo, 1994, 1995a). To understand the equivalence of the model to the first order LWR model, it is necessary to have a look at the traffic states and their units in the dynamics. The original LWR dynamics is:

$$\frac{\partial q(x,t)}{\partial x} = -\frac{\partial \rho(x,t)}{\partial t}$$

or in a difference form:

$$\frac{\Delta(\min\{v\rho, q_{\max}, v(\rho_J - \rho)\})}{\Delta x} = -\frac{\rho(x,t+\Delta t) - \rho(x,t)}{\Delta t}$$

If the equation is discretized by introducing the concept of cell as in Daganzo (1994, 1995a), it can be written as:

$$\rho(i,k+1) - \rho(i,k) = \frac{1}{\bar{v}(i,k)} [Y(i,k) - Y(i+1,k)] \quad (2.1)$$

$$Y_i(i,k) = \min\{v(i,k) \cdot \rho(i,k), q_{\max}, w(i,k) \cdot (\rho_J - \rho(i,k))\}$$

The first equation can be equivalently represented as

$$\rho(i,k+1) = \rho(i,k) + \frac{T}{L} [Y(i,k) - Y(i+1,k)] \quad (2.2)$$

where i is the cell index and k is the time index;

$\rho(i,k)$ – density in cell i at time period k ;

$v(i,k)$ – distance mean speed in cell i at time period k ;

$w(i,k)$ – shockwave back-propagation speed in cell i at time period k ;

q_{\max} – maximum flow of the cell to be identified as a constant parameter – the capacity;

ρ_J – jam density, a known constant parameter.

It is noted that model (1.2) is not a closed dynamical system since it contains speed variable $v(i,k)$ which is not supposed to be in the dynamics although the shockwave speed $w(i,k)$ is known very close a constant for traffic with high enough density. To overcome this difficulty, the Fundamental Diagram (FD) is assumed, which is a static flow-density relationship $q = q(\rho)$ or speed-density relationship $v = v(\rho)$. Once the relationship is modelled for each cell, the first and the third term in the expression of $Y_i(i,k)$ (2.1) can be evaluated.

3. FLOW DYNAMICS FOR CONTROL

Our purpose is to deduce a flow dynamics model suitable for traffic control purpose. First, the flow dynamics is equivalently deduced from LWR model. Then control problem is formulated including the controllability of the model etc.

3.1 Flow Model Deduction

This part deduces the mainline flow model based on LWR model represented as a first order wave equation:

$$\begin{aligned}\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial q(x,t)}{\partial x} &= 0 \\ \rho(x,t) &= \frac{q(x,t)}{v(x,t)}\end{aligned}\quad (3.1)$$

under the assumption that

$$v(x,t) \neq 0$$

In fact, this assumption guarantees that the transformed model is equivalent to the original model (1.1). The case $v(x,t) = 0$ will be treated separately later.

$$\frac{\partial \rho(x,t)}{\partial t} = \frac{\partial \left(\frac{q(x,t)}{v(x,t)} \right)}{\partial t} = \frac{\frac{\partial(q(x,t))}{\partial t} v(x,t) - \frac{\partial v(x,t)}{\partial t} q(x,t)}{v^2(x,t)}$$

Replacing it into the wave equation (1.1), it becomes:

$$\frac{\partial(q(x,t))}{\partial t} v(x,t) - \frac{\partial v(x,t)}{\partial t} q(x,t) = -\frac{\partial q(x,t)}{\partial x} v^2(x,t) \quad (3.2)$$

or equivalently:

$$\frac{\partial(q(x,t))}{\partial t} = \frac{\partial v(x,t)}{\partial t} \frac{q(x,t)}{v(x,t)} - \frac{\partial q(x,t)}{\partial x} v(x,t) \quad (3.3)$$

if $v(x,t) \neq 0$. Now, its corresponding difference equation can be written as

$$\begin{aligned}\frac{q(x,t+\Delta t) - q(x,t)}{\Delta t} &= \\ \frac{[v(x,t+\Delta t) - v(x,t)] q(x,t)}{\Delta t} - \frac{q(x+\Delta x,t) - q(x,t)}{\Delta x} v(x,t)\end{aligned}$$

It can be discretized under the following convention and assumption:

- The cell index is i which represents the cell index with $i+1$ as the index of cell immediately downstream of cell i ;
- The length of cell i is L_i ;
- Speed and flow are constants in each cell;
- Time increment index is k with $t = k \cdot \Delta T$

$$\begin{aligned}q_i(k+1) &= q_i(k) + [v_i(k+1) - v_i(k)] \frac{q_i(k)}{v_i(k)} - \\ &\quad \frac{\Delta T}{L_i} [q_{i+1}(k) - q_i(k)] v_i(k)\end{aligned}\quad (3.4)$$

If we further assume that:

$$\frac{[v_i(k+1) - v_i(k)]}{v_i(k)} \approx \frac{[v_i(k) - v_i(k-1)]}{v_i(k-1)} \quad (3.5)$$

i.e. the previous step relative speed variation is used in place of the next step relative speed variation, it is obtained that

$$\begin{aligned}q_i(k+1) &= \left[1 + \frac{v_i(k) - v_i(k-1)}{v_i(k-1)} \right] q_i(k) - \\ &\quad \frac{\Delta T}{L_i} [q_{i+1}(k) - q_i(k)] v_i(k)\end{aligned}\quad (3.6)$$

Singularity will not happen due to the assumption at the beginning that $v(x,t) \neq 0$ which means that $v_i(k) \neq 0$, for any i and k .

In the case of $v(x,t) = 0$,

$$q(x,t) = 0$$

Thus the following variable structure model is reached:

$$q_i(k+1) = \begin{cases} \left[1 + \frac{v_i(k) - v_i(k-1)}{v_i(k-1)} \right] q_i(k) - \frac{\Delta T}{L_i} [q_{i+1}(k) - q_i(k)] v_i(k), & \text{if } v_i(k-1) \neq 0; \\ 0, & \text{if } v_i(k-1) = 0. \end{cases} \quad (3.7)$$

3.2 To Formulate the Control System

To take into account the interaction between the mainline and the onramp/off-ramp, the first part in model (3.7) should be modified as when $v_i(k-1)$

$$\begin{aligned}q_i(k+1) &= \left[1 + \frac{v_i(k) - v_i(k-1)}{\max\{v_i(k-1), \sigma\}} \right] q_i(k) - \\ &\quad \frac{\Delta T}{L_i} [q_{i+1}(k) - q_i(k)] v_i(k) + r_i(k) - s_i(k)\end{aligned}\quad (3.8)$$

$0 < \sigma \ll 1.0$ is used to avoid singularity. $r_i(k) \geq 0$ is the flow into the mainline from onramp in time step k if applicable, which is a **control variable** for ramp-metering, that the traffic control engineer can manipulate. $s_i(k) \geq 0$ is the flow out of the mainline from off-ramp in time step k , which is to be measured/estimated as a system parameter.

In practical implementation, it is necessary to take into consideration the time interval difference of aggregation for flow and for metering. Suppose the time interval for flow aggregation is ΔT as above; and the metering time interval at the onramp is Δt . Then the following condition is to be satisfied by the onramp flow and metering rate:

$$\begin{aligned}r(k) \cdot \Delta T &= r^{(c)}(k) \cdot \Delta t \\ \text{or } r(k) &= \frac{\Delta t}{\Delta T} \cdot r^{(c)}(k)\end{aligned}\quad (3.9)$$

where $r^{(c)}(k)$ is the practical metering flow rate with respect its own timer. Thus the model (3.8) needs to be modified as

$$q_i(k+1) = \left[1 + \frac{v_i(k) - v_i(k-1)}{\max\{v_i(k-1), \sigma\}} \right] q_i(k) - \frac{\Delta T}{L} [q_{i+1}(k) - q_i(k)] v_i(k) + \frac{\Delta T}{\Delta T} r_i^{(c)}(k) - s_i(k) \quad (3.10)$$

which contains both flow and speed. Therefore it needs further treatment to represent a dynamical system.

4. FLOW MODEL APPLICATION

It is necessary to investigate how the flow dynamics with speed variable involved is used for control. Two possibilities are considered: the first order model and the second order model.

4.1 First Order Model

Since the CTM could be used for ramp metering control Gomes and Horowitz (2006) – to adjust the onramp in-flow rate for controlling the average density in a cells. An immediate question is that can we have a flow dynamics model that is equivalent to the CTM. The answer to this question is negative. The CTM and (3.10) are from the same LRW model. CTM deduction had the assumption of the Fundamental Diagram, i.e. a static relationship between flow and density and therefore a speed-density relationship which is usually monotone decreasing and thus it is well-defined. Based this, the speed variable in the discretized LWR model can be eliminated. However, the speed-flow relationship do not have a inverse as shown in the following figure obtained for Berkeley Highway Lab data (Figure 1). A typical curve fitting of such data will results a curve as in Figure 2, in which one flow will corresponds to two speed values. It is thus concluded that the first order flow dynamics cannot be established in a sensible way as the CTM model.

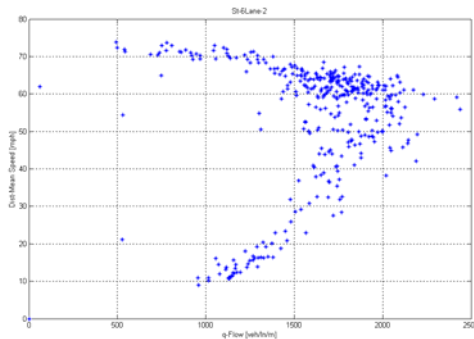


Figure 1. Scatter plot of speed-flow from BHL data, 07/01/2007

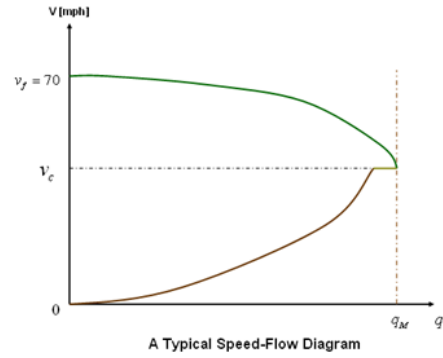


Figure 2, A typical model of speed-flow diagram

4.2 Second Order Model

A feasible application, however, is to add a speed dynamics to the flow dynamics to form a closed second dynamics, such as that in the second order model of Payne (1971), in which the speed dynamics deduction was based on the following assumptions:

- Distance means speed $v(x, t)$ at space-time point (x, t) depends on the density downstream with distance increment Δx ;
- There was a time delay due to driver's response in speed adjustment: $v(x + \tau, t)$ was used instead;
- Driver can predict the traffic speed of the next cell; although this assumption is not necessarily to be true in the past, it is reasonable in a VSL and Coordinated Ramp Metering strategy since traffic situation downstream is taken into account control design. This process implicitly hep the driver to incorporate traffic downstream;
- There is a static relationship between speed and density described as

$$v(x, t + \tau) = V(\rho(x + \Delta x, t)) \quad (4.1)$$

Which could be interpreted as: the density in the $v - \rho$ relationship of the FD has been predicted ahead over distance Δx , but the average driver's response has been delayed by τ in time. Using Taylor expansion to both sides of the equation with respect to t and x respectively and discretize it, one can reach the following speed dynamics.

$$v_i(k+1) = v_i(k) + \frac{T}{\tau} (V(\rho_i(k)) - v_i(k)) - \frac{T}{L} v_i(k) (v_{i-1}(k) - v_i(k)) - \frac{vT}{\tau L} \frac{\rho_{i+1}(k) - \rho_i(k)}{\rho_i(k) + \kappa} \quad (4.2)$$

where T – time step length

L – cell length

τ, ν, κ are parameters to be calibrated from field data.

Each term of the right hand side of the model could be interpreted as follows:

(1) The 1st - relaxation term: It is a high gain filter: the driver is trying to achieve the desired speed $v(\rho_i(k))$ - a **control variable**, where τ is a high gain filter from a dynamic system viewpoint. It can be interpreted as the average response delay of the driver to the desired speed. The definition of the desired speed is critical to reflect the driver behaviour.

(2) The 2nd - convection term: the effect of the traffic into the downstream cell from upstream cell. i.e. the speed increase/decrease caused by in-flow and out-flow vehicle speeds. It can be modified by add a factor $sat\left(\frac{\rho_{i+1}}{\rho_i}\right)$ where $sat(\cdot)$ is the saturation function to address the driver speed change with respect to density variation between the two consecutive cells Cremer and Papageorgiou (1981).

(3) The 3rd - density gradient term: when downstream density increases/decreases, speed in current cell will decrease/increase;

$$\frac{vT}{\tau L} \frac{\rho_{i+1}(k) - \rho_i(k)}{\rho_i(k) + \kappa} = -\frac{1}{\tau} \left(\frac{vT}{L} \frac{\rho_{i+1}(k) - \rho_i(k)}{\rho_i(k) + \kappa} \right)$$

where τ is a high gain filter representing the time delay for the driver's response to the perception of the traffic density (basically, what each driver could observe is the inter-vehicle distance in a immediate vicinity - which could be interpreted as the driver version of *local density*); v is a sensitivity factor. The part in the bracket expresses the effect of downstream cell density: the higher the downstream density, the lower the speed for current cell. $\rho_i(k)$ in the denominator is for normalization. Parameter $\kappa > 0$ is added for two purposes:

- To force the model only work for medium to high density
- To avoid the singularity or abnormal behaviours of the model in low density

The physical meaning of the three terms including the parameters was also explained in details related to driver behaviour in (Cremer and Papageorgiou 1981). Those explanations could be used as the basis for parameter identification.

Since $V(\rho_i(k))$ is basically the speed control parameter to be designed, it could be parameterized with any other value instead of density $\rho_i(k)$ or even without parameterization at all. For example, it could be parameterized with flow $q_i(k)$. Doing this is just a matter of coordinate transformation to understand how the control design would affect the shape changing of FD.

Now (3.2) can be equivalently written as

$$\begin{aligned} v_i(k+1) &= v_i(k) + \frac{T}{\tau} (V - v_i(k)) \\ &- \frac{T}{L} v_i(k) (v_{i+1}(k) - v_i(k)) - \\ &\frac{vT}{\tau L} \left[\frac{q_{i+1}(k) \cdot v_i(k)}{q_i(k) \cdot v_{i+1}(k) + \kappa_1} - 1 \right] \end{aligned} \quad (4.3)$$

where κ_1 is a new system parameter to be calibrated using field data. Its function is to avoid singularity.

Now (3.10) and (4.3) form a well-defined second order dynamical system with state variables $(q_i(k), v_i(k))$ and control parameters $(r_i^{(c)}(k), V)$ - the ramp meter rate and the desired speed limit. It is easy to see that the system is controllable. In practice, it is expected that both control variables need to be subject to some constraints to have physical sense. For example,

$$\begin{aligned} r &\leq r_i^{(c)}(k) \leq \bar{r} \\ 0 &\leq V \leq v_f \end{aligned}$$

It is important to note that the second order model (3.10) and (4.3) are dynamically equivalent to the Payne- Papageorgiou model in principle. This can be observed from the deduction process above.

5. CONCLUSIONS

This paper, from a practical point of view, considers the possibility of using traffic flow instead of density as the state variable for traffic control purpose. The reason to do so is that flow measurement with a point sensor such as inductive loop detector which is dominant sensors used in highway traffic network in California and Nation-wide, can be reasonably accurate but a good density estimation is very costly if not impossible.

The flow-speed-density relationship $q = v\rho$ is used for coordinate transformation in the first order LWR model which is common in almost all first order and second order model - the conservation of vehicle numbers. Based on that, the possibility of establishing a first order model with flow as state variable has been investigated. It turns out that one cannot establish a flow dynamic counterpart of the CTM due to non-existence of the inverse function of the speed-flow relationship - an alternative expression of the Fundamental Diagram.

However, such the flow dynamics coupled with the speed dynamics such as that in the Payne-Papageorgiou model establish an equivalent second order model with flow and speed as state variables. This second order could potentially be used for a combined Variable Speed Limit and Coordinated Ramp Metering design and application with point sensors such as inductive loops and Sensys sensors.

Model analysis for its dynamical behaviour, its validation, simulation using field data, and application to traffic control are the research topics of the next step.

6. ACKNOWLEDGEMENT

This work was supported by the Federal Highway Administration (FHWA) Exploratory Advanced Research Project (Cooperative Agreement DTFH61-07-H-00038) with Match Funding from the California Department of Transportation (Caltrans) TO6224. The contents of this report reflect the views of the authors who are responsible for the facts and the accuracy of the data presented herein. The contents do not necessarily reflect the official views or policies of FHWA and the State of California. This paper does not constitute a standard, specification, or regulation.

REFERENCES

- BHL: <http://www.its.berkeley.edu/bhl/>
- Cremer, M. and Papageorgiou M., (1981). Parameter identification for traffic flow model, *Automatica*, vo. 17, no. 6, p837-843.
- Daganzo, C. F. (1994). The Cell Transmission Model: A Dynamic Representation of Highway Traffic Consistent with the Hydrodynamic Theory. *Transportation Research, B, Vol. 28 (4)*, pp. 269-287.
- Daganzo, C. F., (1995a). The Cell Transmission Model, Part II: Network Traffic. *Transportation Research, B, Vol. 29 (2)*, pp. 79-93.
- Daganzo, C. F., (1995b). Requiem for second-order approximations of traffic flow. *Transportation Research B, Vol. 29 (4)*, p277-286.
- Daganzo, C. F., (1995c). A Finite Difference Approximation of the Kinematic Wave Model of Traffic Flow. *Transportation Research B, Vol. 29 (4)*, pp. 261-276.
- Darbha, S. and Rajagopal, K. R., (2001). Aggregation of a class of linear, interconnected, dynamical systems, *Systems Control Lett.* 43 387-401.
- Gartner, N., Messer, C. J., and Rathi, A. K., (2001). *Traffic Flow Theory, A State-of-the-Art Report*, TRB Committee on Traffic Flow Theory and Characteristics (AHB45), Revised Ed.
- Gomes, G. and Horowitz, R., (2006). Optimal freeway ramp metering using the asymmetric cell transmission model, *Transportation Research Part C*, 14, p244-262
- Hegyi, A., Schutter, B. D. and Hellendoorn, J., (2005). Optimal coordination of variable speed limits to suppress shock waves, *IEEE Trans. on Intelligent Trans. Syst.*, vol. 6, pp. 102-112
- Lighthill M. J. and Whitham G. B. (1955a) On kinematic waves I. Flood movement on long rivers. *Proc. Royal Society of London, Series A*, 229, 281-316.
- Lighthill M. J. and Whitham G. B. (1955b) On kinematic waves II. A theory of traffic flow on long crowded roads. *Proc. Royal Society of London, Series A*, 229, 317-345.
- Lu, X. Y. and Skabardonis, A., (2007). Freeway Traffic Shockwave Analysis: Exploring the NGSIM Trajectory Data, 86th TRB Annual Meeting, Washington, D.C., Paper #07-3016, Jan. 21-25
- Muñoz, L., X. Sun, D. Sun, G. Gomes, and R. Horowitz., (2004). Methodological Calibration of the Cell Transmission Model. In *Proceedings of the 2004 American Control Conference*, Boston, Massachusetts, USA, June 30 – July 2, pp. 798-803.
- Nagel, K., (1998). From particle hopping model to traffic flow theory, TRR1644, Paper number 98-1331, p1-9.
- Newell, G. F., (1993a). A simplified theory of kinematic waves in highway traffic, Part I: General theory, *Trans. Res. B, 27, no. 4*, p281-287
- Newell, G. F., (1993b). A simplified theory of kinematic waves in highway traffic, Part II: Queuing at freeway bottlenecks, *Trans. Res. B, 27, no. 4*, p289-303
- Newell, G. F., (1993c). A simplified theory of kinematic waves in highway traffic, Part III: Multi-destination flows, *Trans. Res. B, 27, no. 4*, p305-313
- Papageorgiou, M., (1983). Applications of Automatic Control Concepts to Traffic Flow Modeling and Control. *Lect. Notes in Control and Inf. Sc.*, Berlin, Germany: Springer
- Papageorgiou, M., Blosseville, J.M., Hadj-Salem, H., (1990). Modelling and real-time control of traffic flow on the southern part of Boulevard Périphérique in Paris. Part I modelling. *Transportation Research A*, 24, 345-359.
- Papamichail, I., Kampitaki, K., Papageorgiou, M., Messmer, A., (2008). Integrated Ramp Metering and Variable Speed Limit Control of Motorway Traffic Flow. Preprints (CD-ROM) of the 17th IFAC World Congress, Seoul, Korea.
- Payne H. J. (1971). Models of freeway traffic and control. *Simulation Council Proc.* 1, 5 1-6 1.
- Richards, P. I. (1956). Shockwaves on the highway. *Operations Research*, 4,42-51.
- Tyagi, V. Darbha, S. and Rajagopal, K. R. (2008). A dynamical systems approach based on averaging to model the macroscopic flow of freeway traffic. *Nonlinear Anal. Hybrid Syst.* 2, no. 2, 590-612
- Zhang, H. M., (1998). A theory of nonequilibrium traffic flow, *Transportation Research, B 32 (7)*, p485-498.
- Zhang, H.M., (1999a). A mathematical theory of traffic hysteresis. *Transportation Research, B 33*, p1-23
- Zhang, H.M., (1999b). Analyses of the stability and wave properties of a new continuum traffic theory. *Transportation Research, B 33 (6)*, p399-415.
- Zhang, H.M., (2000). Structural properties of solutions arising from a nonequilibrium traffic flow theory. *Transportation Research, B 34*, 583-603.