An Equivalent Second Order Model Based on Flow and Speed

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Abstract: Traffic models developed for control include first order LWR model and second order models. The first order model is essentially density dynamics which states the conservation of the number of vehicles in a section of highway. Most second order models are essentially a speed dynamics coupled with the first order density dynamics. There are several second order models but most are for explaining traffic flow. Only a couple of second order models were developed for traffic control purpose. The most popular one is the Payne-Papageorgiou model. Due to point sensor detection limit in highways such as inductive loops, flow estimation is easy and reasonably accurate while density estimation is very difficult. This is the motivation for us to have a look at flow dynamics in place of density. It turns out that flow dynamics of first order cannot be well-defined contrasted to the Cell Transmission Model. However, a flow-speed dynamics can be obtained which is equivalent to the density-speed dynamics of the Payne-Papageorgiou model.

Keywords: traffic flow, density and speed; traffic flow model; first order Cell Transmission Model; second order Payne-Papageorgiou model,

1. INTRODUCTION

In recent years, model-based traffic control design has been becoming more and more popular. The analysis and control design of ramp metering based on the first order Cell Transmission Model (CTM) is one example (Muñoz et al, 2004; Gomes and Horowitz, 2006). Another example is to use a second order model for combined Variable Speed Limit and Coordinated Ramp Meter control design in (Papageorgiou, 1983, 1990), Papamichail et al (2008), and Hegyi (2005). This paper is to addresses some modelling issue based on practical traffic control design considerations.

The CTM is essentially a density dynamics. Thus corresponding control using ramp metering is to control the average density in each cell. For the control design using second order model, the density $\rho$ and distance mean speed $v$ are the state variables. By definition, density is a distance concept which, in principle, can be estimated by the vehicle count instantly captured by video camera for the stretch of highway involved. However, in practice, video camera is not easy to install and maintain. Most traffic sensors in the road are dominantly the inductive loops which provide point measurement. High density loop detector would give a better estimation of density but it is cost prohibitive. It is well-known that loop, particularly a dual loop station, has a good measurement of vehicle count in unit time, or flow. A question naturally arises: is it possible to use flow instead of density as state variable in traffic dynamics for control design? This paper provides an answer to this question. It turns out that the flow cannot be the state in first order model, but can be coupled with speed to form a second order model.

The paper is organized as follows: section 2 is literature review; section 3 deduce the flow dynamics; section 4 discuss the possibility for flow to be a state in first order model, and coupled with speed dynamics to for a second order model; and section 5 is conclusions.

2. LITERATURE REVIEW

There are at least three traffic modelling approaches in the literature for macroscopic traffic modelling.

(1) Based on the physics of fluid flow – traffic is considered as compressible flow. Representative work in this approach is the well-known LWR model Lighthill and Whitham (1955a, b) and Richards (1956), and later development by Newell (1993a, b, c), Daganzo (1995a, b, c) and Zhang (1998, 1999a, 1999b, 2000). A good collection and review of the kinematic wave models and its development history can be found in Gartner et al (2001).

(2) Based on driver behaviour and intuitive understanding of traffic behaviour such as prediction and delay. Representative work in this direction is the model by Payne (1971) and the improvement by Papageorgiou (1983), which is called Payne-Papageorgiou model here. It is essentially a second model with coupled density and speed dynamics. Since then, several second order models have been obtained by some modification/improvement on the model. The model has been further improved in (Papageorgiou, 1990) by introducing the
waving effect due to lane changing. Since the authors intended for ramp meter control and different ramp meter strategy evaluation only, there was essentially only one control variable – the ramp meter rate. The speed dynamics is not intended to be independent. Instead, it generated a reference speed based a static density-speed relationship – the Fundament Diagram and went back to the loop to affect the density dynamics indirectly through some coupling. In this sense, the second model is essentially a dynamics of density with some driver behaviour added through the speed dynamics. The theoretical model of second order was used in (Hegyi et al, 2005) for combined VSL and Ramp Metering for reducing shockwave. A good reference is referred to (Nagel, 1998) for understanding several second order traffic model based on fluid dynamics. The explanation is very interesting, particularly the car following model and the Payne- Papageorgiou model.

3. FLOW DYNAMICS FOR CONTROL

Our purpose is to deduce a flow dynamics model suitable for traffic control purpose. First, the flow dynamics is equivalently deduced from LWR model. Then control problem is formulated including the controllability of the model etc.

3.1 Flow Model Deduction

This part deduces the mainline flow model based on LWR model represented as a first order wave equation:

\[ \frac{\partial q(x,t)}{\partial x} = -\frac{\partial (x,t)}{\partial t} \]

or in a difference form:

\[ \frac{\Delta}{\Delta x} \min \{ v(\rho, q_{\text{max}}) \} = \rho(x,t+\Delta t) - \rho(x,t) \]

If the equation is discretized by introducing the concept of cell as in Daganzo (1994, 1995a), it can be written as:

\[ \rho(i,k+1) - \rho(i,k) = \frac{1}{\tau(i,k)} \left[ Y(i,k) - Y(i+1,k) \right] \]

\[ Y(i,k) = \min \{ v(i,k) \cdot \rho(i,k), q_{\text{max}}, w(i,k) \cdot (\rho_j - \rho(i,k)) \} \]

The first equation can be equivalently represented as

\[ \rho(i,k+1) = \rho(i,k) + \frac{T}{L} [Y(i,k) - Y(i+1,k)] \]

where \( i \) is the cell index and \( k \) is the time index; \( \rho(i,k) \) – density in cell \( i \) at time period \( k \); \( v(i,k) \) – distance mean speed in cell \( i \) at time period \( k \); \( w(i,k) \) – shockwave back-propagation speed in cell \( i \) at time period \( k \); \( q_{\text{max}} \) – maximum flow of the cell to be identified as a constant parameter – the capacity; \( \rho_j \) – jam density, a known constant parameter.

It is noted that model (1.2) is not a closed dynamical system since it contains speed variable \( v(i,k) \) which is not supposed to be in the dynamics although the shockwave speed \( w(i,k) \) is known very close a constant for traffic with high enough density. To overcome this difficulty, the Fundamental Diagram (FD) is assumed, which is a static flow-density relationship \( q = q(\rho) \) or speed-density relationship \( v = v(\rho) \). Once the relationship is modelled for each cell, the first and the third term in the expression of \( Y(i,k) \) (2.1) can be evaluated.
\[
\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial q(x,t)}{\partial x} = 0
\]
under the assumption that \( v(x,t) \neq 0 \).

In fact, this assumption guarantees that the transformed model is equivalent to the original model (1.1). The case \( v(x,t) = 0 \) will be treated separately later.

Replacing it into the wave equation (1.1), it becomes:

\[
\frac{\partial (q(x,t))}{\partial t} v(x,t) - \frac{\partial v(x,t)}{\partial t} q(x,t) = -\frac{\partial q(x,t)}{\partial x} v(x,t)
\]

or equivalently:

\[
\frac{\partial (q(x,t))}{\partial t} v(x,t) - \frac{\partial v(x,t)}{\partial t} q(x,t) = -\frac{\partial q(x,t)}{\partial x} v(x,t)
\]

if \( v(x,t) \neq 0 \). Now, its corresponding difference equation can be written as

\[
q_i(k+1) = q_i(k) + \left[ \frac{v_i(k+1) - v_i(k)}{\Delta t} \right] q_i(k) - \left[ \frac{q_i(k+1) - q_i(k)}{\Delta x} \right] v_i(k)
\]

\[
q_i(k+1) = \left[ 1 + \frac{v_i(k) - v_i(k-1)}{v_i(k-1)} \right] q_i(k) - \frac{\Delta T}{L_i} [q_{i-1}(k) - q_i(k)] v_i(k)
\]

Thus the following variable structure model is reached:

\[
q_i(k+1) = \left[ 1 + \frac{v_i(k) - v_i(k-1)}{v_i(k-1)} \right] q_i(k) - \frac{\Delta T}{L_i} [q_{i-1}(k) - q_i(k)] v_i(k)
\]

3.2 To Formulate the Control System

To take into account the interaction between the mainline and the onramp/off-ramp, the first part in model (3.7) should be modified as when \( v_i(k-1) \neq 0 \)

\[
q_i(k+1) = \left[ 1 + \frac{v_i(k) - v_i(k-1)}{v_i(k-1)} \right] q_i(k) - \frac{\Delta T}{L_i} [q_{i-1}(k) - q_i(k)] v_i(k)
\]

\[
q_i(k+1) = \left[ 1 + \frac{v_i(k) - v_i(k-1)}{v_i(k-1)} \right] q_i(k) - \frac{\Delta T}{L_i} [q_{i-1}(k) - q_i(k)] v_i(k) + r_i(k) - s_i(k)
\]

where \( 0 < \sigma \ll 1.0 \) is used to avoid singularity. \( r_i(k) \geq 0 \) is the flow into the mainline from onramp in time step \( k \) if applicable, which is a control variable for ramp-metering, that the traffic control engineer can manipulate. \( s_i(k) \geq 0 \) is the flow out of the mainline from off-ramp in time step \( k \), which is to be measured/estimated as a system parameter.

In practical implementation, it is necessary to take into consideration the time interval difference of aggregation for flow and for metering. Suppose the time interval for flow aggregation is \( \Delta T' \) as above; and the metering time interval at the onramp is \( \Delta t \). Then the following condition is to be satisfied by the onramp flow and metering rate:

\[
r_i(k) - \Delta t' r_i(k) = \Delta t r_i(k) - \frac{\Delta T}{L_i} [q_{i-1}(k) - q_i(k)] v_i(k)
\]

If we further assume that:

\[
\left[ \frac{v_i(k+1) - v_i(k)}{\Delta t} \right] \approx \left[ \frac{v_i(k) - v_i(k-1)}{\Delta t} \right]
\]

i.e. the previous step relative speed variation is used in place of the next step relative speed variation, it is obtained that

\[
\frac{\Delta x}{\Delta t} = \frac{v_i(k)}{v_i(k-1)}
\]

\[
q_i(k+1) = \frac{v_i(k)}{v_i(k-1)} q_i(k)
\]

\[
q_i(k+1) = \frac{v_i(k)}{v_i(k-1)} q_i(k)
\]
which contains both flow and speed. Therefore it needs further treatment to represent a dynamical system.

4. FLOW MODEL APPLICATION

It is necessary to investigate how the flow dynamics with speed variable involved is used for control. Two possibilities are considered: the first order model and the second order model.

4.1 First Order Model

Since the CTM could be used for ramp metering control Gomes and Horowitz (2006) – to adjust the onramp in-flow rate for controlling the average density in a cells. An immediate question is that can we have a flow dynamics model that is equivalent to the CTM. The answer to this question is negative. The CTM and (3.10) are from the same LRW model. CTM deduction had the assumption of the Fundamental Diagram, i.e. a static relationship between flow and density and therefore a speed-density relationship which is usually monotone decreasing and thus it is well-defined. Based this, the speed variable in the discretized LWR model can be eliminated. However, the speed-flow relationship do not have a inverse as shown in the following figure obtained for Berkeley Highway Lab data (Figure 1). A typical curve fitting of such data will results a curve as in Figure 2, in which one flow will corresponds to two speed values. It is thus concluded that the first order flow dynamics cannot be established in a sensible way as the CTM model.

![Figure 1. Scatter plot of speed-flow from BHL data, 07/01/2007](image)

![Figure 2. A typical model of speed-flow diagram](image)

4.2 Second Order Model

A feasible application, however, is to add a speed dynamics to the flow dynamics to form a closed second dynamics, such as that in the second order model of Payne (1971), in which the speed dynamics deduction was based on the following assumptions:

- Distance means speed \( v(x, t) \) at space-time point \((x,t)\) depends on the density downstream with distance increment \( \Delta x \);
- There was a time delay due to driver’s response in speed adjustment: \( v(x + \tau, t) \) was used instead;
- Driver can predict the traffic speed of the next cell; although this assumption is not necessarily to be true in the past, it is reasonable in a VSL and Coordinated Ramp Metering strategy since traffic situation downstream is taken into account control design. This process implicitly hep the driver to incorporate traffic downstream;
- There is a static relationship between speed and density described as

\[
v(x, t + \tau) = V(x + \Delta x, t)
\]  

Where could be interpreted as: the density in the \( v - \rho \) relationship of the FD has been predicted ahead over distance \( \Delta x \), but the average driver’s response has been delayed by \( \tau \) in time. Using Taylor expansion to both sides of the equation with respect to \( t \) and \( x \) respectively and discretize it, one can reach the following speed dynamics.

\[
v_i(k+1) = v_i(k) + \frac{T}{\tau} (\Delta v_i(k) - v_i(k)) - \frac{T}{\Delta L} \rho_i(k) - \rho_i(k) - \frac{\kappa}{\tau L}
\]  

where \( T \) – time step length
\( L \) – cell length
\( \tau, \nu, \kappa \) are parameters to be calibrated from field data.

Each term of the right hand side of the model could be interpreted as follows:
(1) The 1st - relaxation term: It is a high gain filter: the driver is trying to achieve the desired speed \( V(\rho(k)) \) - a control variable, where \( \tau \) is a high gain filter from a dynamic system viewpoint. It can be interpreted as the average response delay of the driver to the desired speed. The definition of the desired speed is critical to reflect the driver behaviour.

(2) The 2nd - convection term: the effect of the traffic into the downstream cell from upstream cell. i.e. the speed increase/decrease caused by in-flow and out-flow vehicle speeds. It can be modified by add a factor \( \text{sat}(\rho_i/\rho) \) where \( \text{sat}(\cdot) \) is the saturation function to address the driver speed change with respect to density variation between the two consecutive cells Cremer and Papageorgiou (1981).

(3) The 3rd - density gradient term: when downstream density increases/decreases, speed in current cell will decrease/increase:

\[
\frac{\nu T \rho_{in}(k) - \rho(k)}{\tau L \rho(k) + \kappa} = -1 \left( \frac{\nu T \rho_{in}(k) - \rho(k)}{\tau L \rho(k) + \kappa} \right)
\]

where \( \tau \) is a high gain filter representing the time delay for the driver’s response to the perception of the traffic density (basically, what each driver cold observe is the inter-vehicle distance in a immediate vicinity - which could be interpreted as the driver version of local density); \( \nu \) is a sensitivity factor. The part in the bracket expresses the effect of downstream cell density: the higher the downstream density, the lower the speed for current cell. \( \rho(k) \) in the denominator is for normalization. Parameter \( \kappa > 0 \) is added for two purposes:

- To force the model only work for medium to high density
- To avoid the singularity or abnormal behaviours of the model in low density

The physical meaning of the three terms including the parameters was also explained in details related to driver behaviour in (Cremer and Papageorgiou 1981). Those explanations could be used as the basis for parameter identification.

Since \( V(\rho(k)) \) is basically the speed control parameter to be designed, it could be parameterized with any other value instead of density \( \rho(k) \) or even without parameterization at all. For example, it could be parameterized with flow \( q_i(k) \).

Doing this is just a matter of coordinate transformation to understand how the control design would affect the shape changing of FD.

Now (3.2) can be equivalently written as

\[
v_i(k+1) = v_i(k) + \frac{T}{\tau}(V - v_i(k))
\]

\[
The equation above is the second order model (3.10) and (4.3) are dynamically equivalent to the Payne-Papageorgiou model – the conservation of vehicle numbers. Based on that, the possibility of establishing a first order model with flow as state variable has been investigated. It turns out that one cannot establish a flow dynamic counterpart of the CTM due to non-existence of the inverse function of the speed-flow relationship – an alternative expression of the Fundamental Diagram.

However, such the flow dynamics coupled with the speed dynamics such as that in the Payne-Papageorgiou model establish an equivalent second order model with flow and speed as state variables. This second order could potentially be used for a combined Variable Speed Limit and Coordinated Ramp Metering design and application with point sensors such as inductive loops and Sensys sensors.

Model analysis for its dynamical behaviour, its validation, simulation using field data, and application to traffic control are the research topics of the next step.
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