

Macroscopic Traffic Flow Stability for Adaptive Cruise Controlled (ACC) Vehicles

Jingang Yi* and Roberto Horowitz

Department of Mechanical Engineering, University of California, Berkeley, CA 94720

Abstract

In this paper, we discuss a general approach to the macroscopic traffic flow stability analysis of adaptive cruise controlled (ACC) vehicles. We present a macroscopic model with velocity saturation for traffic flow in which each individual vehicle is controlled by an ACC spacing policy. A nonlinear traffic flow stability criterion is investigated using a wavefront expansion technique. Qualitative relationships between traffic flow stability and model parameters are derived for a generalized ACC traffic flow model. The newly derived stability results are in agreement with previously derived results that were obtained using both microscopic and macroscopic models with a constant time headway (CTH) policy. Moreover, the stability results derived in this paper provide sufficient and necessary conditions for ACC traffic flow stability, which can be used to design other ACC spacing policies.

1 Introduction

Traffic flow stability is an important subject because congestion caused by an unstable traffic stream degrades the performance of road transportation networks. Recently, a new class of so-called adaptive cruise control (ACC), or autonomous intelligent cruise control (AICC) system has been developed for vehicles operating in manual traffic, which are able not only to maintain a constant velocity but also, in addition, either a constant distance or a constant headway from the preceding vehicle. The stability of traffic flow, when vehicles are operating under ACC, has received increasing attention by the IVHS and AHS community, and this is the topic of this paper.

In [1] stability of traffic flow under an ACC spacing policy was first studied, using an aggregated macroscopic traffic flow model, for an open stretch highway. Using a linearized stability analysis, they showed that the traffic flow equilibrium state was marginally stable under a constant time headway (CTH) policy. The conclusions in [2] seemed to contradict the results in [1] at a first glimpse. Instead of studying an open stretch highway,

[2] studied a circular highway, in order to eliminate the entry and exit effects on the intrinsic stability property of the ACC CTH policy. Analyses based on a microscopic model, a spatially discrete model and a spatially continuous model were discussed and several stability conclusions were obtained, depending on the choice of aggregating biasing strategy that was used in abstracting the highway's macroscopic dynamics. The ACC system dynamics were not considered in [2] and the authors claimed that the stability property of traffic flow should be robust to such unmodeled dynamics. Recently, the results in [3] provided an explanation of the discrepancy between the results in [1] and [2].

In this paper, we discuss a framework to study the traffic flow stability using a general continuous traffic flow model for vehicles under ACC spacing policies. An intrinsic stability criterion of the ACC traffic flow systems is presented using a wavefront expansion technique [4, 5]. The stability criterion is independent of the particular ACC policy used, and the stability conditions derived are both necessary and sufficient. The stability results for the CTH policy are investigated and found to be consistent with all previous results obtained in [1], [2] and [3]. One attractive property of the proposed stability analysis technique is that it can be used to design and validate other ACC policies with more precision than prior techniques.

This paper consists of five parts. In section 2, a macroscopic ACC traffic flow model is presented and a definition of stability of traffic flow is presented. The main results are presented in section 3 and traffic stability criteria are analyzed using a wavefront expansion technique. Comparisons and discussion for the CTH policy are presented in section 4. Concluding remarks are presented in section 5.

2 Macroscopic ACC Traffic Flow Model and Stability

In this paper we consider only a one lane highway without any on- or off-ramps.¹ Let $\rho(x, t)$ denote the high-

¹The inlet and outlet flows are considered as the on- and off-ramps. We can drop this assumption by adding relaxation terms

*Corresponding author. Email: jgyi@gandhi.me.berkeley.edu.

way density, $q(x, t)$ the flow rate and $v(x, t)$ the traffic velocity at position x along the highway at time t , respectively. By definition, $q(x, t) = \rho(x, t)v(x, t)$. Note that the traffic velocity $v(x, t)$ is the average or aggregated speed of all individual vehicles on the highway around x . Conservation of vehicles on the highway gives us the following equation ²:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0 \quad (1)$$

and the velocity dynamics can be written, in a general form, as

$$v(x, t+T) = \begin{cases} v_f & 0 \leq \rho(x+\Delta, t) \leq \rho_{min}, \\ h(\rho(x+\Delta, t)) & \rho_{min} \leq \rho(x+\Delta, t) \leq \rho_{max}, \end{cases} \quad (2)$$

where T is the relaxation time of the ACC system dynamics, v_f is the highway free flow velocity, Δ is the spatially biasing strategy distance, $\rho_{min} = h^{-1}(v_f)$ and ρ_{max} is the maximum highway flow ³.

The underlying physical meaning given by Eq. (2) is that the speed of the traffic flow at time $t+T$ is regulated as the traffic concentration policy $h(\rho)$ at position $x+\Delta$ at the time t . If we consider an ideal ACC system as in [2], then $T=0$. For the spatial biasing strategy, we use the same notation as in [2]. If $\Delta(x, t) > 0$, it is a downstream biasing; if $\Delta(x, t) = 0$, it is a neutral biasing; and if $\Delta(x, t) < 0$, it is an upstream biasing. In [1], only the neutral biasing, $\Delta(x, t) = 0$, was considered.

The spacing policy $h(\rho)$ is similar to the velocity-density fundamental diagram (v - ρ curve) of manual traffic flow. We consider a general function form $h(\rho)$ of this policy. As in the v - ρ fundamental diagram for manual traffic flow, the spacing policy function $h(\rho)$ of ACC traffic flow satisfies

$$h'(\rho) = \frac{dh}{d\rho} \leq 0. \quad (3)$$

For example, under the CTH policy, $h(\rho)$ is given by

$$h(\rho) = \begin{cases} v_f & 0 \leq \rho(x, t) \leq \rho_{min}, \\ \frac{1}{h_w} \left(\frac{1}{\rho} - L_v \right) & \rho_{min} < \rho(x, t) \leq \rho_{max}, \end{cases} \quad (4)$$

where h_w is the constant time headway and L_v is the average vehicle length. In this case, $\rho_{min} = \frac{1}{v_f h_w + L_v}$ and $\rho_{max} = \frac{1}{L_v}$. A schematic of traffic velocity and flow rate under the CTH policy is shown in Fig. 1. The above CTH policy (4) was used in [1], [2] and [3].

in right hand side of the conservation laws.

²In this paper we assume that the traffic flow variables $\rho(x, t)$, $q(x, t)$ and $v(x, t)$ are differentiable for simplicity.

³The highway maximum flow density can be simply calculated by $\frac{1}{L_v}$, where L_v is the average vehicle length, i.e. the bumper-to-bumper flow density.

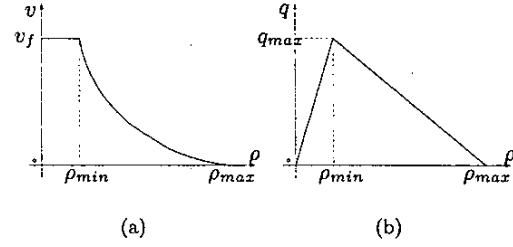


Figure 1: A schematic of the CTH policy (a) velocity-density relationship (v - ρ curve); (b) flow-density relationship (q - ρ curve).

Using Taylor's expansion and neglecting higher-order terms, from Eq. (2), we obtain

$$\frac{\partial v}{\partial t} - \mu h'(\rho) \frac{\partial \rho}{\partial x} = -\frac{1}{T} (v - h(\rho)), \quad (5)$$

where $\mu := \frac{\Delta(x, t)}{T}$. In this paper, we consider μ is a constant for simplicity.

Consider now the ACC traffic flow dynamics given by Eqs. (1) and (5). In this paper, we discuss the stability of ACC traffic flow under perturbed initial conditions. The definition of the traffic flow propagation stability, which this paper focus on, is given as follow [5].

Definition 1 (Traffic flow propagation stability)

Let $\mathbf{q}_e(x, t) = [\rho_e(x, t), v_e(x, t)]^T$ denote the nominal equilibrium state of the traffic system on a highway with length L . Let $\mathbf{q}_p(x, t)$ be the perturbed state. The traffic flow $\mathbf{q}_e(x, t)$ is **propagation stable** under perturbation traffic state \mathbf{q}_p if the spatial gradient of the perturbed state is bounded, i.e. $\|\frac{\partial \mathbf{q}_p}{\partial x}(x, t)\| < \infty$, for $\forall t > 0, x \in [0, L]$. If, in addition to the above, $\lim_{t \rightarrow \infty} \|\frac{\partial \mathbf{q}_p}{\partial x}(x, t)\| = 0$, then, the traffic state \mathbf{q}_e is **asymptotically propagation stable**.

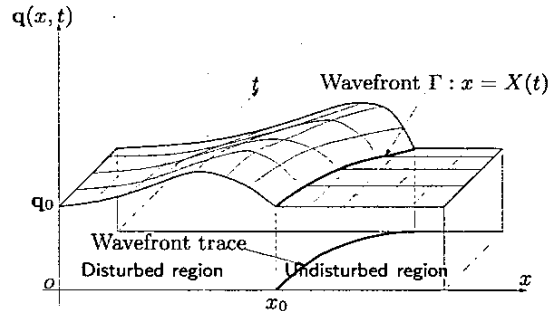


Figure 2: A schematic of a solution surface, wavefront and wavefront trace with an initial C^1 discontinuous condition $q(x_0, 0)$.

In [1], [2] and [3], the asymptotic stability was analyzed using the stability theory of systems governed by ordinary differential equations (ODE), i.e. microscopic and spatially discrete models. For a system governed by partial differential equations, if we consider that a perturbation happens on a smooth density or velocity profile and if the perturbation attenuates to the equilibrium state, we normally consider the system to be asymptotically stable. Therefore, if the traffic flow system satisfies the propagation stability and in addition $\|\frac{\partial q_0}{\partial x}\| \rightarrow 0$ as $t \rightarrow \infty$, we will consider that the system is asymptotically stable in this paper.

The wavefront of a traffic system can be illustrated by Fig. 2 as a separation curve between disturbed and undisturbed regions. In this paper, we consider that the undisturbed region of the solution surface is flat for simplicity. Moreover, the initial condition of the system (1) and (2), $q_0(x, 0)$, is not smooth (C^∞) at x_0 . For the traffic system, it is easy to see that the wavefront and wavefront trace can be expressed as the same equation in the $x-t$ plane.

To discuss traffic flow stability we need to first investigate the characteristic velocity of the traffic system given by Eqs. (1) and (5). From now on, we use shortened notation to denote partial derivatives, for example, $\rho_x := \frac{\partial \rho}{\partial x}$. Following the same approach as in [5], we obtain the following expression for the characteristic velocities v_c

$$v_c := \frac{dx}{dt} = \frac{v \pm \sqrt{v^2 - 4\mu\rho h'(\rho)}}{2}. \quad (6)$$

3 Stability Conditions

3.1 Wavefront expansion

In this section we discuss the propagation stability conditions for an ACC traffic flow model (Eqs. (1) and (5)) under large perturbations. Suppose that traffic has a constant flow density ρ_0 and velocity $v_0 = h(\rho_0)$. Obviously, v_0, ρ_0 are the solutions of Eqs. (1) and (5). Near the wavefront, we can use the time variable t as a parameter and write the equation of the wavefront in the form $x = X(t)$. Using the same approach that was used to analyze manual traffic in [5], it is particularly convenient to expand the solution of the system around the wavefront in powers of

$$\zeta = x - X(t) \quad (7)$$

where the wavefront has the characteristic velocity v_c at the equilibrium states, i.e.

$$\dot{X}(t) = v_c(\rho_0, v_0) = v_0 + u_0, \quad (8)$$

$$\text{where } u_0 := \frac{-v_0 \pm \sqrt{v_0^2 - 4\mu\rho_0 h'(\rho_0)}}{2}.$$

Using Eq. (7), we can expand the flow variables ρ and v behind the wavefront in a power series of ζ as

$$\rho(x, t) = \rho_0 + \zeta\rho_1(t) + \frac{1}{2}\zeta^2\rho_2(t) + \dots \quad (9)$$

$$v(x, t) = v_0 + \zeta v_1(t) + \frac{1}{2}\zeta^2 v_2(t) + \dots, \quad (10)$$

where

$$\rho_i(t) = \left. \frac{\partial^i \rho}{\partial x^i} \right|_{(X(t)^-, t)}, \quad v_i(t) = \left. \frac{\partial^i v}{\partial x^i} \right|_{(X(t)^-, t)}, \quad i = 1, 2, 3, \dots$$

We now calculate the partial derivatives of state variables ρ and v , using Eqs. (9) and (10),

$$\rho_t = -\dot{X}(t)\rho_1(t) + \zeta\dot{\rho}_1(t) + \zeta \left[-\dot{X}(t) \right] \rho_2(t) + \dots \quad (11a)$$

$$\rho_x = \rho_1(t) + \zeta\rho_2(t) + \frac{1}{2}\zeta^2\rho_3(t) + \dots \quad (11b)$$

$$v_t = -\dot{X}(t)v_1(t) + \zeta\dot{v}_1(t) + \zeta \left[-\dot{X}(t) \right] v_2(t) + \dots \quad (11c)$$

$$v_x = v_1(t) + \zeta v_2(t) + \frac{1}{2}\zeta^2 v_3(t) + \dots \quad (11d)$$

$$h(\rho) = h(\rho_0) + \zeta h'(\rho_0)\rho_1 + \frac{1}{2}\zeta^2 h''(\rho_0) + \dots \quad (11e)$$

$$h'(\rho) = h'(\rho_0) + \zeta h''(\rho_0)\rho_1 + \frac{1}{2}\zeta^2 h'''(\rho_0) + \dots \quad (11f)$$

Substituting all of the above equations into the flow dynamics and arranging terms into successive powers of ζ , we obtain

$$\dot{v}_1 + \alpha v_1 + \beta v_1^2 = 0, \quad (12)$$

where

$$\alpha = \frac{1}{T} \frac{\rho_0 u_0 h'(\rho_0) - u_0^2}{\rho_0 \mu h'(\rho_0) - u_0^2} = \frac{1}{T} \frac{u_0 - \rho_0 h'(\rho_0)}{2u_0 + v_0},$$

$$\beta = \frac{\rho_0 \mu [2h'(\rho_0) + \rho_0 h''(\rho_0)]}{\rho_0 \mu h'(\rho_0) - u_0^2}. \quad (13)$$

Notice that $v_1(t) = \left. \frac{\partial v(x, t)}{\partial x} \right|_{(X(t)^-, t)}$, namely, the slope of wavefront at point P . The above equation gives the slope evolution at the wavefront. The propagation stability of Eq. (12) can thus be analyzed in terms of the initial condition $v_1(0)$ and the parameters α and β . Table 1 shows the stability conditions of the system given by Eq. (12).

3.2 Stability conditions

In this section, we discuss the stability conditions for different choice of the parameter $\mu = \frac{\Delta(x, t)}{T}$, i.e. different relaxation time T and biasing distance Δ , and the ACC policy $h(\rho)$.

Table 1: Stability conditions for dynamical system Eq. (12)

Parameters α and β	Stable region
$\beta > 0, \alpha > 0$	$v_1(0) \in [-\frac{\alpha}{\beta}, \infty), v_1(t) \rightarrow 0$
$\beta > 0, \alpha = 0$	$v_1(0) \in \mathbb{R}, v_1(t) \rightarrow 0$
$\beta > 0, \alpha < 0$	$v_1(0) \in [0, \infty), v_1(t) \rightarrow -\frac{\alpha}{\beta}$
$\beta < 0, \alpha > 0$	$v_1(0) \in (-\infty, -\frac{\alpha}{\beta}), v_1(t) \rightarrow 0$
$\beta < 0, \alpha = 0$	\emptyset
$\beta < 0, \alpha < 0$	$v_1(0) \in (-\infty, 0), v_1(t) \rightarrow -\frac{\alpha}{\beta}$
$\beta = 0, \alpha > 0$	$v_1(0) \in \mathbb{R}, v_1(t) \rightarrow 0$
$\beta = 0, \alpha = 0$	$v_1(0) \in \mathbb{R}, v_1(t) \equiv v_1(0)$
$\beta = 0, \alpha < 0$	\emptyset

(1). $T > 0, \Delta > 0$.

In this case, $\mu = \frac{\Delta}{T} > 0$, and we have

- If $h'(\rho_0) = 0$, i.e. under uncongested traffic flow and each vehicle travels at its maximum velocity v_f , we obtain

$$\alpha = \frac{1}{T} > 0, \quad \beta = 0.$$

From Table 1, we can conclude that the system is asymptotically stable for any initial condition of the perturbations.

- If $h'(\rho_0) < 0$, we can choose

$$u_0 = \frac{-v_0 - \sqrt{v_0^2 - 4\rho_0\mu h'(\rho_0)}}{2} < 0,$$

because of $v_c = v_0 + u_0 < v_0$ and the fact that it is realistic to keep the perturbation wave propagating upstream along the traffic flow. Let $\delta := \sqrt{v_0^2 - 4\rho_0\mu h'(\rho_0)} > 0$ and $u_0 = -\frac{v_0 + \delta}{2}$. Thus, we obtain

$$\alpha = \frac{\rho_0 h'(\rho_0) - u_0}{T\delta}, \quad \beta = \frac{\rho_0\mu [2h'(\rho_0) + \rho_0 h''(\rho_0)]}{u_0\delta}. \quad (14)$$

From Table 1, in order to achieve asymptotic stability for any initial condition $v_1(0) \geq -\frac{\alpha}{\beta}$, we need (i) $\beta > 0, \alpha > 0$, or (ii) $\beta > 0, \alpha = 0$, or (iii) $\beta = 0, \alpha > 0$. Before we discuss each case in detail, it is interesting to calculate the conditions on $h(\rho)$ to attain stability when $\alpha = 0$ or $\beta = 0$.

If $\alpha = 0$, we obtain a first-order ODE of the function $h(\rho)$

$$\rho h'(\rho) + h(\rho) + \mu = 0.$$

Solving the above equation with the physical constrained that $h\left(\frac{1}{L_v}\right) = 0$, we obtain

$$h(\rho) = \frac{\mu}{L_v\rho} - \mu = \frac{\mu}{L_v} \left(\frac{1}{\rho} - L_v \right), \quad (15)$$

which is exactly the CTH policy given by Eq. (4) with a constant headway time $h_w = \frac{L_v}{\mu}$. Moreover, if we apply the same calculation when $\beta = 0$, we can find that the spacing policy function $h(\rho)$ must satisfy

$$h(\rho) = \frac{c_1}{\rho} + c_2, \quad (16)$$

where c_1 and c_2 are constants determined by the constrained conditions imposed on $h(\rho)$. Not surprisingly, the function form given by Eq. (16) is the same as Eq. (4). Thus, we can conclude that, if spacing policy $h(\rho)$ satisfies the conditions that $\alpha = 0$, then it must also satisfy $\beta = 0$. However, the converse is not always true because it depends on whether $h_w = \frac{L_v}{\mu}$.

The above analysis rules out case (ii) because, if $\alpha = \beta = 0$, the system is marginally stable but not asymptotically stable. We need to discuss the other two cases separately. For case (i), $\beta > 0, \alpha > 0$, we need

$$\begin{cases} \left. \frac{dq}{d\rho} \right|_{\rho_0} + \mu > 0 \\ \left. \frac{d^2q}{d\rho^2} \right|_{\rho_0} < 0 \end{cases} \quad (17)$$

The second condition given in (17) reminds us of the concave condition for the flow-density (q - ρ) relationship in manual traffic flow [6]. In this case, when we design the ACC policy, we still need to satisfy this condition for ACC traffic flow. The first condition in (17) implies that the slope of the q - ρ curve must be greater than $-\mu$.

For case (iii), $\beta = 0, \alpha > 0$, it is easy to see that the ACC policy must be the CTH given by Eq. (16) and, under such a CTH policy, the constant time headway constant h_w must be chosen such that

$$\frac{L_v}{h_w} < \mu, \quad (18)$$

in order to guarantee the condition $\alpha > 0$.

Summary: In the case when $T > 0, \Delta > 0$, i.e. $\mu > 0$, the ACC traffic flow is asymptotically stable for an equilibrium state iff

- (i) traffic flow travels at velocity v_f ;
- (ii) the CTH policy is used with that the constant headway time h_w is chosen such that $\frac{L_v}{h_w} < \mu$;
- (iii) for other ACC spacing policies, if the proposed flow-density relationship, q - ρ curve, is concave and its slope is larger than $-\mu$ at this equilibrium state.

(2). $T > 0, \Delta = 0$. In this case $\mu = 0$, and thus $\beta = 0$. In order to guarantee asymptotic stability for any initial condition of $v_1(0)$, from Table 1 we need $\alpha > 0$. However, we have $u_0 = 0$ or $u_0 = -v_0$. However,

note that $u_0 = 0$ is impossible because, if $u_0 = 0$, we have that $v_1(t) = 0$, which is not true for the perturbed velocity profile. Thus, $u_0 = -v_0$, and from (13) we have

$$\alpha = \frac{1}{T} \frac{\rho_0 u_0 h'(\rho_0) - u_0^2}{-u_0^2} = \frac{1}{T v_0} \frac{dq}{d\rho} \Big|_{\rho_0}.$$

The stability condition is then

$$h'(\rho_0) = 0, \text{ or } \frac{dq}{d\rho} \Big|_{\rho_0} > 0. \quad (19)$$

Summary: In the case when $T > 0$, $\Delta = 0$, i.e. $\mu = 0$, the ACC traffic flow is asymptotically stable for an equilibrium state iff

- (i) traffic flow travels at velocity v_f ;
- (ii) under the ACC spacing policy, the slope of the proposed flow-density relationship, q - ρ curve, is positive at this equilibrium state.

- (3). $T > 0$, $\Delta < 0$. In this case $\mu < 0$, similarly to the first case when $\mu > 0$, if $h'(\rho_0) = 0$, i.e. an uncongested traffic flow condition, the system is asymptotically stable; if $h'(\rho_0) < 0$. Then

$$u_0 = \frac{-v_0 \pm \sqrt{v_0^2 - 4\rho_0 \mu h'(\rho_0)}}{2} = \frac{-v_0 \pm \delta}{2} < 0,$$

where $\delta = \sqrt{v_0^2 - 4\rho_0 \mu h'(\rho_0)}$. We can follow the same analysis as in the first case of $\mu > 0$, except that, in this case, the values of u_0 can take two values. We can thus have the stability conditions:

$$\begin{cases} \frac{dq}{d\rho} \Big|_{\rho_0} + \mu > 0, \\ \frac{d^2q}{d\rho^2} \Big|_{\rho_0} \neq 0. \end{cases} \quad (20)$$

For the case when $\beta = 0$, $\alpha > 0$, this implies the CTH policy and both choices of u_0 give the same condition:

$$-\frac{L_v}{h_w} + \mu > 0,$$

which is never satisfied for $\mu < 0$!

Summary: In the case when $T > 0$, $\Delta < 0$, i.e. $\mu < 0$, the ACC traffic flow is asymptotically stable for an equilibrium state iff,

- (i) traffic flow travels at velocity v_f ;
- (ii) under the ACC spacing policy, if the proposed flow-density relationship, q - ρ curve, is either concave or convex and its slope is larger than $-\mu$ at this equilibrium state;

- (3). $T = 0$ and $\Delta \neq 0$. In this case $\mu = \infty$. We can apply the results given in the previous cases by letting $T \rightarrow 0$ for each case of Δ .

- $\Delta > 0$, $T \rightarrow 0$ implies $\mu \rightarrow +\infty$. From case (1) discussed above, it is easy to obtain that: (1) the ACC traffic flow is asymptotically stable for the CTH policy; (2) for any ACC spacing policy, if the q - ρ relationship is concave at an equilibrium state, then the state is asymptotically stable; (3) any uncongested traffic equilibrium state, i.e. traffic flow travels at velocity v_f , is asymptotically stable.

- $\Delta < 0$, $T \rightarrow 0$ implies $\mu \rightarrow -\infty$. From case (3) discussed above, it is easy to conclude that the system is asymptotically stable only if for an uncongested traffic equilibrium state, i.e. traffic flow travels at velocity v_f .

- (4). $\Delta = 0$ and $\Delta = 0$, in this case, the system (1) and (5) reduces to a first-order nonlinear PDE system, similar to the LWR model [7], which is stable in the sense of Lyapunov, or marginally stable. Any disturbance will propagate along the wavefront without any amplifications or attenuation [4].

4 Comparisons and Discussion

In this section, we compare the results presented in the previous section with those reported in [1], [2] and [3]. Here we only consider a dense traffic equilibrium density $\rho_0 > \rho_{min}$ ⁴.

In [1], the authors considered the ACC system dynamics on the traffic flow by a first-order regulator with an exponential decaying rate $\frac{1}{\tau}$. The traffic flow dynamics are given in PDE form as

$$\begin{cases} \frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0, \\ \frac{\partial v}{\partial t} + (v + \rho h'(\rho)) \frac{\partial v}{\partial x} = -\frac{1}{\tau}(v - h(\rho)). \end{cases} \quad (21)$$

Comparing these dynamics with the traffic flow dynamics (1) and (5) discussed in this paper, we notice that the first equation is same. One difference between the second equation in (21) and Eq. (5), i.e. the velocity dynamics, is that we accounted for the spatial biasing strategy by introducing Δ for the ACC policy, while in [1] no such consideration exists for a spatially continuous model. Thus, the velocity dynamics in (21) have no density gradient term $\frac{\partial \rho}{\partial x}$, which appeared in (5). Another difference comes from the method for abstracting the ACC system dynamics. In [1], the authors took the total derivative of traffic flow velocity, $\dot{v}(x, t) = v \frac{\partial v}{\partial x} + \frac{\partial v}{\partial t}$, as the controlled *vehicle* acceleration; while in this paper we consider the partial derivative of traffic flow with time, $\frac{\partial v(x, t)}{\partial t}$, as the controlled acceleration of *traffic flow* at a fixed position x .

Even though there exists minor difference in the system dynamics, the stability conditions derived in this

⁴When a traffic equilibrium density $\rho_0 \leq \rho_{min}$, it is always asymptotically stable for any ACC spacing policies.

paper are consistent with those found in [1]. In [1], a linearized stability analysis was discussed for the PDE system and the authors only concluded that, when $\tau \rightarrow 0$, system (21) behaves like the LWR model for manual traffic systems, i.e. marginally stable (or stable in the sense of Lyapunov). If we consider $\Delta = 0$ and let $T \rightarrow 0$, we obtain the same conclusion.

We can further investigate the stability conditions for the systems given by (21). We can consider a more general velocity dynamics of the traffic flow of ACC vehicles by modifying the dynamics (2) for dense flow as follows

$$v(x(t+T), t+T) = h(\rho(x(t+T) + \Delta, t+T)). \quad (22)$$

In the above velocity dynamics, we consider that the vehicle velocity of ACC traffic flow at position $x(t+T)$ and time $t+T$ is regulated by the density at position $x(t+T) + \Delta$ and time $t+T$. The difference between the dynamics (2) and (22) is that in (22) we take the position variable x to be changing with time t in the microscopic level (individual vehicle) instead of traffic flow velocity at a fix position x in the macroscopic level (as in Eq. (2)). We can expand Eq. (22) using Taylor series and conservation law (1) and obtain

$$\frac{\partial v}{\partial t} + (v + \rho h') \frac{\partial v}{\partial x} - \mu h' \frac{\partial \rho}{\partial x} = -\frac{1}{T} (v - h(\rho)), \quad (23)$$

where $\mu = \frac{\Delta}{T}$. It is easy to see that when $\Delta = 0$, i.e. $\mu = 0$, Eq. (23) reduces to Eq. (21). For the hyperbolic system (1) and (23), we can apply the same process of the wavefront expansion technique to check the stability conditions. The Riccati equation is the same as Eq. (12) with

$$\alpha = \frac{u_0^2 \frac{h''(\rho_0)}{h'(\rho_0)} - \frac{1}{T} \mu}{u_0(u_0 - 2\mu)}, \quad \beta = \frac{2u_0 - 3\mu - \rho_0 \mu \frac{h''(\rho_0)}{h'(\rho_0)}}{u_0 - 2\mu}, \quad (24)$$

where $u_0 = \frac{1}{2} \left[\rho_0 h' \pm \sqrt{(\rho_0 h')^2 - 4\mu \rho_0 h'} \right]$. If we consider the case of the ACC CTH policy given by [1], then $\mu = 0$, $h(\rho) = \frac{1}{h_w} \left(\frac{1}{\rho} - L_v \right)$, and we obtain

$$\alpha = -\frac{2}{\rho_0} < 0, \quad \beta = 2 > 0. \quad (25)$$

From Table 1, we can conclude that system is marginal stable and moreover, the slope of the disturbance along the wavefront converges to a constant $-\frac{\alpha}{\beta} = \frac{1}{\rho_0}$, which is consistent with the result obtained by [1].

In [2], the authors discussed the stability of a circular traffic flow under an ACC CTH policy (4). A spatially biasing strategy was used to model the ACC spacing policy. The spatially continuous model in [2] is given by

$$\begin{cases} \frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0, \\ v(x, t) = \frac{1}{h_w} \left(\frac{1}{\rho(x+\Delta(x,t), t)} - L_v \right), \end{cases} \quad (26)$$

where $\Delta(x, t)$ could be positive, zero or negative, depending on whether the spatially biasing strategy is downstream, neutral and upstream, respectively. The velocity dynamics in (26) do not consider the ACC system dynamics and the author assumed that "an infinite vehicle speed can be achieved". In [2], the stability conclusions for various biasing strategies are: for downstream biasing ($\Delta > 0$), system is asymptotically stable; for neutral biasing ($\Delta = 0$), system is marginally stable; and for upstream biasing ($\Delta < 0$), system is unstable. Note that, if we let $T = 0$ for the model (1) and (5) used in this paper, we obtain exactly the same model (26) given in [2]. Therefore, applying $T = 0$ for the stability results in the previous section and the ACC CTH policy (4), we reach the same stability conclusion.

In [3], the authors did not use a spatially continuous PDE model to investigate stability. Instead, a spatially biasing discrete model was used. The authors concluded that, when $\frac{dq}{d\rho} > 0$ at the equilibrium state, then the traffic flow was stable for all of boundary conditions, or so-called unconditional traffic flow stable. The authors applied this stability criterion to the neutral biasing ACC CTH policy and a conclusion of instability was reached because of $\frac{dq}{d\rho} = -\frac{L_v}{h_w} < 0$. Moreover, the authors suggested an alternative variable time-gap (VTG) spacing policy under which the traffic flow was unconditionally stable within a critical density value. The VTG policy proposed in [3] is given by

$$v = v_f \left(1 - \frac{\rho}{\rho_{max}} \right), \quad (27)$$

where v_f is the free velocity and ρ_{max} is the maximum density. Note that the VTG policy given by Eq. (27) is the same as the fundamental diagram in manual traffic flow.

In [3], the authors used the neutral biasing strategy to check the stability of the ACC CTH policy. The stability condition, $\frac{dq}{d\rho} > 0$, is consistent with the results we obtained under $T > 0$ and $\Delta = 0$. However, when [3] proved the stability condition $\frac{dq}{d\rho} > 0$, an upstream biasing strategy, $0.5 < \alpha_i \leq 1$, was used in the spatially discrete model⁵. In this case, the stability condition resulting from our analysis is $\left. \frac{dq}{d\rho} \right|_{\rho_0} > |\mu|$, which is different from the condition $\frac{dq}{d\rho} > 0$ given by [3].

In order to investigate this discrepancy further, we take the proposed VTG policy in [3] as an example, and apply the two stability criteria. Fig 3(a) shows the flow-density

⁵The biasing strategy in the spatially discrete model is determined by magnitude of the weighting coefficient α_i , $0 \leq \alpha_i \leq 1$, which is used in calculating the traffic flow rate q_i crossing the i^{th} section, $q_i = \alpha_i \rho_i v_i + (1 - \alpha_i) \rho_{i+1} v_{i+1}$. If $0 \leq \alpha_i < 0.5$, downstream biasing; $\alpha_i = 0.5$, neutral biasing; $0.5 < \alpha_i \leq 1$, upstream biasing.

relationship (q - ρ curve) resulting from Eq. (27). Notice that

$$q = \rho v = v_f \left(\rho - \frac{\rho^2}{\rho_{max}} \right) \implies \frac{dq}{d\rho} = v_f \left(1 - \frac{2\rho}{\rho_{max}} \right)$$

The stability condition given in [3] tells us that, if the traffic density $\rho_0 \in [0, \frac{\rho_{max}}{2}]$, then ρ_0 is asymptotically stable; while the stability condition in this paper states that if the traffic density $\rho_0 \in [0, \rho_c]$, then ρ_0 is asymptotically stable, where $\rho_c = \frac{\rho_{max}}{2} \left(1 - \frac{|\mu|}{v_f} \right) < \frac{\rho_{max}}{2}$ for $\mu \neq 0$. Notice that at the density $\frac{\rho_{max}}{2}$, the traffic system has its maximum flow rate q_m , and the stability condition $\frac{dq}{d\rho} > 0$ tells that, before the traffic flow rate reaches the maximum value q_m , it is asymptotically stable (Fig. 3(a)). This is not a case from observations of the real traffic flow data. Fig. 3(b) shows the flow-density relationship at one spot on the Southern California I-210 on April 25, 2001. By curve fitting Eq. (27), we obtain that $\rho_{max} = 88$ (veh/mile/lane) and $v_f = 60$ mph. From the data, we can roughly see that before the traffic flow reaches its maximum volume q_m (≈ 2200 veh/hour/lane for highway I-210) and critical density $\rho_{max}/2$ (≈ 44 veh/mile/lane), the traffic density becomes unstable around a critical density ρ_c (≈ 33 veh/mile/lane) and jumps to the unstable branch. The data in Fig. 3(b) is from a manual traffic flow; however, since the VTG policy (27) is also used to model the manual traffic velocity-density relationship, we believe that a similar behavior will be observed for ACC vehicles controlled under the spacing policy (27).

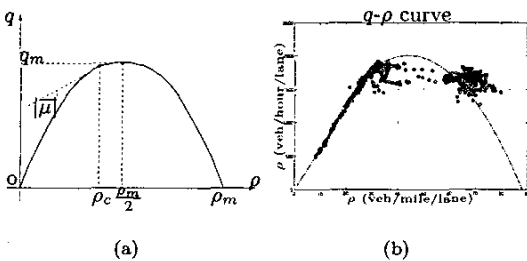


Figure 3: Flow-density relationship (q - ρ curve) (a) VTG policy as Eq. (27); (b) traffic data of highway I-210 (dot) and fitting curve (solid).

If we consider the stability condition $\frac{dq}{d\rho} > 0$ in [3] for downstream biasing strategies, we can find that it is a sufficient condition. In this case, $T > 0$ and $\Delta > 0$, we can find that the conditions for traffic asymptotic stability are $\frac{dq}{d\rho} + \mu > 0$ and $\frac{d^2q}{d\rho^2} \leq 0$. For the first condition, it is automatically satisfied if $\frac{dq}{d\rho} > 0$ with the condition $\mu = \frac{\Delta}{T} > 0$.

5 Conclusions

In this paper we discussed the traffic flow propagation stability of a highway system where all vehicles are controlled by an adaptive cruise control (ACC) spacing policy using a nonlinear stability analysis technique. A generalized macroscopic traffic model with velocity saturation for the ACC traffic flow system was proposed. A generalized stability criterion for an asymptotically stable traffic flow system was derived using a wavefront expansion method. We discussed and compared this stability criterion with those previously obtained by other authors. We found that the stability results derived in this paper covered all stability conditions obtained for ACC spacing policies in addition to the constant time headway (CTH) policy. Moreover, the nonlinear stability analysis in this paper provided more precise stability information under perturbations than the previous approximate linearized stability approach. The stability condition derived in this paper can be used for prediction of ACC traffic flow stability, and to perform a qualitative stability analysis as well as to design other ACC spacing policies.

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