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Observer based emergency braking control in automated highway systems*

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Abstract

Emergency braking maneuver control in automated highway systems (AHS) is addressed. Based on the on-line estimation of the longitudinal velocity of the vehicle and the tire/road friction characteristics, a pressure in the master cylinder of the braking system that attempts to achieve maximum deceleration during braking is calculated. Based on the estimated characteristics a braking strategy is applied. The designed system provides information for safe vehicle spacing and traffic flow control.

1 Introduction

Automated Highway Systems (AHS) is a concept introduced to address increasing highway capacity demands without compromising safety levels. Emergency braking maneuvers are expected to be necessary for safe fault handling in AHS [1]. The AHS longitudinal feedback maneuvers designed in [2] that incorporate emergency braking are proven to be safe if a maximum on the braking capabilities of vehicles can be established. This capability changes with adverse environmental conditions, gradual wear of components and highway topology, etc., and is mainly determined by two factors: tire/road friction and available braking torque. These factors have complex behavior and the associated variables are difficult to measure.

The longitudinal velocity of vehicles is normally calculated based on either the angular velocity of the wheels or the engine velocity. This can not be done during emergency braking, as the slip between the tires and road is no longer negligible. Another calculation procedure is required.

The goal of this paper is to design a controller for emergency braking of vehicles based on the on-line estimation of the vehicle's longitudinal velocity and the tire/road friction characteristics. Knowledge of the tire/road friction characteristics allows vehicles to adjust their spacing for safety and broadcast this information to the road-side infrastructure. The roadside controller can then modify overall traffic conditions if necessary.

This paper builds upon the work presented in [3, 4] that introduced the tire/road friction model used here and a controller for emergency braking. In those papers it was assumed that the longitudinal velocity of the vehicle was known. This assumption is relaxed in this paper and replaced with the assumption that the longitudinal acceleration is known, which may be easier to fulfill in practice, by utilizing an accelerometer.

2 Vehicle Modeling

A quarter vehicle model is used to describe the vehicle longitudinal dynamics. The longitudinal motion of the vehicle is expressed by

$$m\dot{v} = 4F_x - F_{ax}\,, \tag{1}$$

where v is the longitudinal speed of the vehicle, m is the vehicle mass, F_x is the force at the tire/road interface and F_{ax} is the aerodynamic drag force. It is assumed that forces at the tire/road interfaces are evenly distributed. The rotational dynamics at

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the wheel are described by

$$J\dot{\omega} = \tau_d - \tau_b - F_x R, \quad (2)$$

where ω is the angular velocity, J the wheel inertia, τ_d the driving torque, τ_b the braking torque and R the effective rolling radius. Eqs. (1) and (2) assume that the longitudinal velocity v and the wheel angular velocity ω are related through the relative velocity, s, defined as

$$s = v - R\omega \,. \quad (3)$$

Relative velocity s and slip λ during braking are related by $\lambda = s/v$. The drag force and tire force are defined by

$$F_{ax} = C_{ax}v^2 \,; \quad F_x = -\mu F_N = -\mu\frac{mg}{4} \,, \quad (4)$$

with C_{ax} a properly chosen constant, μ the tire/road friction coefficient and $F_N = mg/4$ the normal load in each tire.

Substituting Eqs. (4) into Eqs. (1) and (2) and using the time derivative of Eq. (3) yields

$$\dot{v} = -c\,\mu - d\,v^2 \,, \quad (5)$$

$$\dot{s} = -(a+c)\,\mu - b - dv^2 + e\,K_b P_b \,, \quad (6)$$

with $a = R^2 mg/4J$, $b = R\tau_d/J$, $c = g$, $d = C_{ax}/m$ and $e = R/J$. As suggested in [5], the braking torque is approximated by $\tau_b = K_b P_b$, where K_b is an overall braking system gain and P_b the master cylinder pressure. $\tau_d = 0$ is assumed during the braking process. The velocity v and relative velocity s will be assumed to be uniformly continuous functions.

3 Tire/road Friction Characteristics

Pseudo-static tire/road models describe the coefficient of friction, μ, as a function of the wheel slip λ and additional parameters like the speed and normal load [6, 7]. Previous work on real time estimation of μ is presented, for example, in [8], [9] and [10][1]. The model proposed in [7] is

$$\mu = \left(C_1(1 - e^{-C_2\lambda_u}) - C_3\lambda_u\right) e^{-C_4 v} \,, \quad (7)$$

[1] For a more extended revision of the existing literature the reader is referred to [3].

where $\lambda_u = |\lambda|$, $C_1, \cdots, C_4$ are constants and the normal load at the tire is kept constant. In this paper this model is approximated by

$$\mu = p_1 e^{-p_2\lambda_u} \lambda_u^{(p_3\lambda_u + p_4)} e^{-p_5 v} \,, \quad (8)$$

where p_1, p_2, p_3, p_4 and p_5 are parameters to be determined. After applying a logarithm to both sides and rearranging in vector form, Eq. (8) becomes

$$y = [1,\ -\lambda_u,\ \lambda_u \ln\lambda_u,\ \ln\lambda_u,\ -v] \begin{bmatrix} p_1' \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{bmatrix} = \mathbf{U\Theta} \,, \quad (9)$$

where $p_1' = \ln p_1$. An estimate $\hat{\Theta}$ of the vector Θ will be proposed in a following section.

If the velocity is kept constant, the peak value of μ_m can be derived from Eq. (8),

$$\mu_m\Big|_{v=v_0} = p_1 e^{-p_2\lambda_m} \lambda_m^{(p_3\lambda_m + p_4)} e^{-p_5 v_0} \,, \quad (10)$$

where λ_m, the peak slip is given by the solution to

$$p_3\lambda_m(\ln\lambda_m + 1) = p_2\lambda_m - p_4 \,. \quad (11)$$

Notice that the peak slip is not dependent on the velocity for the model proposed in Eqs. (7) and (8).

4 Observer-Controller design

To design the emergency braking controller, it is first necessary to design an observer for the longitudinal velocity. It will be assumed that measurements of angular velocity and longitudinal acceleration are available, that the angular acceleration can be derived from the angular velocity and that the overall gain of the braking system is known.

For the estimation of the longitudinal velocity the following structure is proposed

$$\dot{\hat{v}} = -\,c\,\mu - d\,\hat{v}^2 + L\,\tilde{y}_2 \quad (12)$$

where $\hat{v}$ is the estimated value for v. The coefficient of friction μ is deduced from

$$\mu = -\frac{J\dot{\omega} + eK_b P_b}{f} \,, \quad (13)$$

with $f = mgR/4$, $\tilde{y}_2$ is given by

$$\begin{aligned} \tilde{y}_2 &= \dot{v} - \dot{\hat{v}} = (-c\mu - dv^2) - (-c\mu - d\hat{v}^2) \\ &= -\,d\tilde{v}(v + \hat{v}) \end{aligned} \quad (14)$$

and L is a gain to be determined. Notice that the definition of $\hat{v}$ is different from that of $\hat{v}$ introduced in Eq. (12).

From Eqs. (5) and (12) the dynamics of the velocity estimation error $\tilde{v} = v - \hat{v}$ is

$$\dot{\tilde{v}} = -dv^2 + d\hat{v}^2 - L\tilde{y}_2 = -d\tilde{v}(v + \hat{v})(1 - L) \,, \tag{15}$$

An important property of Eq. (15) is that the sign of $\tilde{v}$ does not change. This property is stated in the following Lemma.

Lemma 1 *If $\tilde{v}(0) < 0$ and $L < 0$ then $\tilde{v}(t) < 0 \; \forall t \geq 0$.*

Proof: For any given value of v and $\hat{v}$ the solution to Eq. (15) is of form

$$\tilde{v}(t) = \tilde{v}(0)e^{-(1-L)\int_0^t d(v+\hat{v})d\tau} \tag{16}$$

This term will never change sign, therefore $\tilde{v}(0) < 0 \Rightarrow \tilde{v}(t) < 0 \; \forall t \geq 0$. ■

Define the following Lyapunov function candidate

$$W_a = \frac{1}{2}\tilde{v}^2 \tag{17}$$

Taking the time derivative of Eq. (17) and using Eq. (15)

$$\dot{W}_a = \tilde{v}\dot{\tilde{v}} = -d\tilde{v}^2(v + \hat{v})(1 - L), \tag{18}$$

Choosing $L < 0$ guarantees that

$$\dot{W}_a \leq -\alpha_a \phi_a^2(\tilde{v}) \leq 0, \tag{19}$$

with $\alpha_a = 2d(1 - L)v_{min} > 0$ and $\phi_a(\tilde{v}) = \tilde{v}$. v_{min} is a bound on the longitudinal velocity of the vehicle such that the emergency braking maneuver can be considered complete when $v \to v_{min}$. The exponential stability of $\tilde{v} = 0$ follows.

To continue with the controller design, it is necessary to set the value for the pressure of the master cylinder, P_b; for that purpose define

$$\bar{s} = \hat{s} - \hat{\lambda}_m \hat{v} = \hat{v}(1 - \hat{\lambda}_m) - R\omega \tag{20}$$

as the desired relative velocity for the emergency braking maneuver. In this expression $\hat{s} = \hat{v} - R\omega$ and $\hat{\lambda}_m$ is the estimated value of λ_m based on the current estimation of $\hat{\mathbf{\Theta}}$. Taking the time derivative of Eq. (20)

$$\dot{\bar{s}} = \dot{\hat{v}}(1 - \hat{\lambda}_m) - R\dot{\omega} - \hat{v}\dot{\hat{\lambda}}_m = \dot{\hat{v}}(1 - \hat{\lambda}_m) - \frac{Rf}{J}\mu + \frac{ReK_bP_b}{J} - \hat{v}\frac{\partial\hat{\lambda}_m}{\partial\hat{v}}\dot{\hat{v}} - \hat{v}\frac{\partial\hat{\lambda}_m}{\partial\omega}\dot{\omega} \,. \tag{21}$$

The partial derivatives of λ_m can be calculated from Eq. (11). Choosing

$$P_b = \frac{J}{ReK_b}\Big(-\dot{\hat{v}}(1 - \hat{\lambda}_m) + \frac{Rf}{J}\mu + \hat{v}\frac{\partial\hat{\lambda}_m}{\partial\hat{v}}\dot{\hat{v}} + \hat{v}\frac{\partial\hat{\lambda}_m}{\partial\omega}\dot{\omega} - \zeta\bar{s}\Big) \tag{22}$$

and substituting in Eq. (21) gives

$$\dot{\bar{s}} = -\zeta\bar{s} \,. \tag{23}$$

Define the following Lyapunov function candidate

$$W_b = \frac{1}{2}\bar{s}^2 \tag{24}$$

Taking the time derivative of Eq. (24) and using Eq. (23)

$$\dot{W}_b \leq -\alpha_b\phi_b^2(\bar{s}) \leq 0, \tag{25}$$

with $\alpha_b = \zeta > 0$ and $\phi_b(\bar{s}) = \bar{s}$. The exponential stability of $\bar{s} = 0$ follows.

The next step in the design of the observer-controller is to analyze the behavior of the estimation of the tire/road friction and the vehicle velocity. Define now the following Lyapunov function candidate

$$W_c = \frac{1}{2}v^2 \tag{26}$$

Taking the time derivative of Eq. (26) and using Eq. (5)

$$\dot{W}_c = -v(c\mu + dv^2) \leq -\alpha_c\phi_c^2(v) \leq 0, \tag{27}$$

with $\alpha_c = d > 0$ and $\phi_c(v) = v^{3/2} \geq 0$. The exponential stability of $v = 0$ follows.

Consider the following gradient type parameter adaptation law (PAA) for the tire/road model

$$\dot{\hat{\mathbf{\Theta}}} = -\Gamma\hat{\mathbf{U}}^T\tilde{y} \tag{28}$$

where $\tilde{y} = \ln\mu - \ln\hat{\mu} = \mathbf{U\Theta} - \hat{\mathbf{U}}\hat{\mathbf{\Theta}}$.

Define the last Lyapunov function candidate

$$W_d = \frac{1}{2}\tilde{\Theta}^T\Gamma^{-1}\tilde{\Theta} + \frac{1}{2}\tilde{v}^2 \tag{29}$$

Taking the time derivative of Eq. (29) and using Eqs. (15) and (28)

$$\begin{aligned}\dot{W}_d &= \hat{v}(-c\mu - dv^2 + Ldv^2 - Ld\hat{v}^2) \\ &\quad - \tilde{\Theta}^T\hat{\mathbf{U}}^T(\mathbf{U}\Theta - \hat{\mathbf{U}}\hat{\Theta} + \hat{\mathbf{U}}\Theta - \hat{\mathbf{U}}\Theta) \\ &\leq -\alpha_d\phi_d^2(\hat{v}) - \alpha_e\phi_e^2(\tilde{\Theta}) + Ldv^2\hat{v} + \tilde{\Theta}^T\hat{\mathbf{U}}^T\tilde{\mathbf{U}}\Theta\end{aligned} \tag{30}$$

where $\tilde{\mathbf{U}} = \mathbf{U} - \hat{\mathbf{U}}$, $\alpha_d = d(1-L) > 0$, $\alpha_e = K_e$, $\phi_d(\hat{v}) = \hat{v}$ and $\phi_e(\tilde{\Theta}) = \sqrt{\tilde{\Theta}^T\tilde{\Theta}}$. The value of K_e satisfies $K_e \leq \|\ln \lambda_{min}\|$, with λ_{min} a prescribed acceptable minimum value for λ during the braking process.

The last two terms in Eq. (30) can not be guaranteed to be negative semi-definite. To deal with them a result from [11] will be used. First it will be shown in the appendix that for all $v \geq v_{min}$

$$\tilde{\mathbf{U}}\Theta \leq K_1|\tilde{v}| \tag{31}$$

with v_{min} a small velocity introduced to avoid singularities when in the braking process $v \to 0$, and that

$$Ldv^2 \leq \gamma_{cd}\phi_c(v) \tag{32}$$

with $\gamma_{cd} = Ldv_{max}^{\frac{1}{2}}$. Using Eqs. (31) and (32), Eq. (30) can be expressed as

$$\begin{aligned}\dot{W}_d \leq &-\alpha_d\phi_d^2(\hat{v}) - \alpha_e\phi_e^2(\tilde{\Theta}) + \gamma_{cd}\phi_c(v)\phi_d(\hat{v}) \\ &+ \gamma_{ae}\phi_a(\tilde{v})\phi_e(\tilde{\Theta})\end{aligned} \tag{33}$$

with $\gamma_{ae} = K_1$.

Define the composite Lyapunov function candidate

$$W = \sum_{i\in\{a,b,c,d\}} d_i W_i\ , \tag{34}$$

where $d_i > 0$ are scaling factors to be determined. The time derivative of Eq. (34) satisfies, from Eqs. (19),(25), (27) and (33)

$$\begin{aligned}\dot{W} \leq &- d_a\alpha_a\phi_a^2(\tilde{v}) - d_b\alpha_b\phi_b^2(\bar{s}) - d_c\alpha_c\phi_c^2(v) \\ &+ d_d(-\alpha_d\phi_d^2(\hat{v}) + \gamma_{cd}\phi_c(v)\phi_d(\hat{v})) \\ &+ d_e(-\alpha_e\phi_e^2(\tilde{\Theta}) + \gamma_{ae}\phi_a(\tilde{v})\phi_e(\tilde{\Theta}))\end{aligned} \tag{35}$$

Eq. (35) is a quadratic form that can be rewritten as

$$\dot{W} \leq -\frac{1}{2}\Phi^T\left(\mathbf{DS} + \mathbf{S}^T\mathbf{D}\right)\Phi \tag{36}$$

where $\Phi = [\phi_a, \cdots, \phi_e]^T$, $\mathbf{D} = diag\,(d_a, \cdots, d_e)$ and the matrix $\mathbf{S}$ is defined by

$$s_{ij} = \begin{cases} \alpha_i & i = j \\ -\gamma_{ij} & i \neq j \end{cases} \tag{37}$$

where $\gamma_{ij} = 0,\ \ \forall ij \neq cd, ae$.

The quadratic form in Eq. (36) is negative definite if and only if the matrix $\mathbf{DS} + \mathbf{S}^T\mathbf{D}$ is positive definite. The following Lemma, taken from [11], guarantees the existence of the matrix $\mathbf{D}$ if some conditions in $\mathbf{S}$ are satisfied.

Lemma 2 *There exists a positive diagonal matrix* $\mathbf{D}$ *such that* $\mathbf{DS} + \mathbf{S}^T\mathbf{D}$ *is positive definite if and only if* $\mathbf{S}$ *is an M-matrix; that is, the leading principal minors of* $\mathbf{S}$ *are positive.*

It is straightforward to prove that the matrix $\mathbf{S}$ has positive principal minors. This proves the asymptotic stability of the equilibrium $\tilde{v} = v = \hat{v} = \bar{s} = 0$. For the adapted parameters this result only shows that $\tilde{y} = 0$, which does not necessarily imply $\tilde{\Theta} = 0$, if there is no persistence of excitation.

In practice control gains are tuned to obtain the scaling factors d_i.

5 Simulation Results

Simulation of an emergency braking maneuver using the compensator introduced in the previous section was performed. The results are shown in Figs. 1-4. Fig. 1 contains plots with the evolution of the estimated parameters. Fig. 2 shows the braking torque and deceleration attained during the maneuver. Fig. 3 displays the evolution of the state for the observer and controller. Fig. 4 illustrates the final estimate of the λ vs. μ curve. The plots indicate good performance of the controller. The "true parameters" for the approximation in Eq. (8) were obtained with an off-line test and are shown in Table 1 together with the real value of the braking system gain. The trajectory imposed by emergency braking does not guarantee parameter convergence, however, simulation results show errors in parameter convergence that are below 10%.

The simulation results show that the conditions obtained in [4] for guaranteeing the underestimation of the maximum friction coefficient μ_{max} do not hold when the state observer is introduced. How-

Table 1: Parameters for the approximation in Eq. (8)

p_1	p_2	p_3	p_4	p_5	K_b
3.16	3.3	2.64	1.05	0.01	0.9

ever simulation results indicate that the underestimation of λ_m is still preserved[2]. This is illustrated in Fig. 4, which shows the reference tire/road friction curve at 15 m/s, along with the approximation obtained, shown as the curve that does not cover the full range of slip. An analysis similar to the one in [4], to determine there exist conditions in the parameter estimate initial conditions and adaptation gains, which perserve the properties obtained in [4] is in progress.

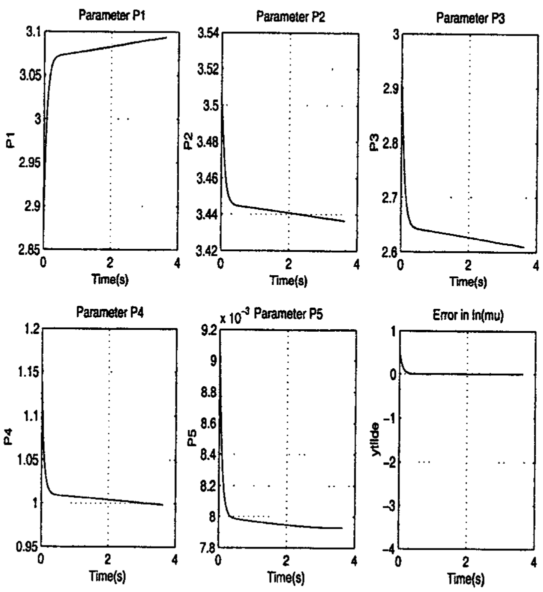

Figure 1: Simulation results of emergency braking: adapted parameters and $\tilde{y}$.

6 Conclusions

A controller for emergency braking maneuvers of vehicles is designed. This controller estimates the longitudinal velocity and tire/road characteristics and attempts to achieve maximum braking effort during the entire maneuver. Velocity dependence of the tire/road friction is explicitly considered.

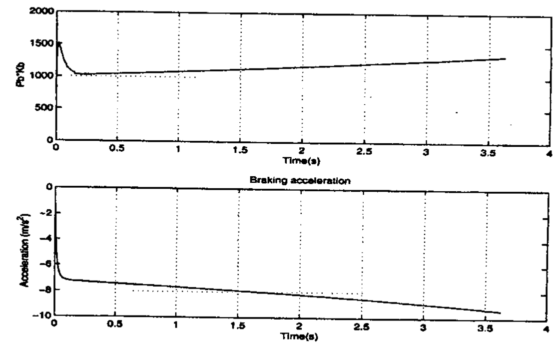

Figure 2: Simulation results of emergency braking: braking torque and deceleration.

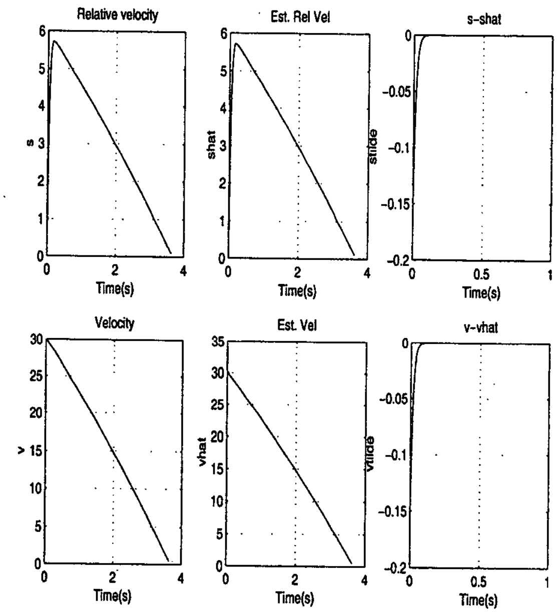

Figure 3: Simulation results of emergency braking: state evolution and error signals.

The stability of the controller is proven and simulation results in accordance with the theoretical findings are included. The use of this controller avoids the chattering effect presented by ABS systems and provides a source of information for on-line safe spacing calculations.

This controller achieves good results even when there is no parameter convergence; in this case the results are not optimal. The underestimation of λ_m appears in simulations. The analytical proof of this important property is ongoing work.

[2]More details of this underestimation property can be found in [4].

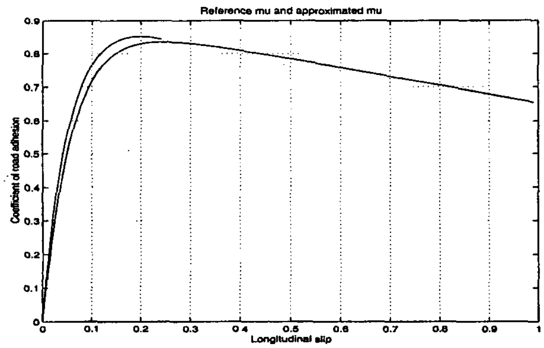

Figure 4: Simulation results of emergency braking; final λ vs. μ curve. The adapted curve is truncated.

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Appendix: Bound on the term $\tilde{\mathbf{U}}\Theta$

First notice that by assumption Θ is bounded. The other term, $\tilde{\mathbf{U}}$ is

$$\begin{aligned}\tilde{\mathbf{U}} &= [1, -\lambda, \lambda\ln\lambda, \ln\lambda, -v] - [1, -\hat{\lambda}, \hat{\lambda}\ln\hat{\lambda}, \ln\hat{\lambda}, \hat{v}] \\ &= [0, -\lambda+\hat{\lambda}, \lambda\ln\lambda - \hat{\lambda}\ln\hat{\lambda}, \ln\lambda - \ln\hat{\lambda}, -\tilde{v}] .\end{aligned} \tag{38}$$

The second element in $\tilde{\mathbf{U}}$ can be expressed as

$$-\lambda+\hat{\lambda} = -1 + \frac{R\omega}{v} + 1 - \frac{R\omega}{\hat{v}} = -\frac{R\omega}{v\hat{v}}\tilde{v} . \tag{39}$$

The third element in $\tilde{\mathbf{U}}$ is reduced to

$$\begin{aligned}\lambda\ln\lambda - \hat{\lambda}\ln\hat{\lambda} &= \frac{v-R\omega}{v}(\ln(v-R\omega) - \ln v) \\ &\quad - \frac{\hat{v}-R\omega}{\hat{v}}(\ln(\hat{v}-R\omega) - \ln\hat{v}) \\ &= -\frac{R\omega}{v(v-R\omega)}\tilde{v} + -R\omega^2\frac{v+\hat{v}-R\omega}{v(v-R\omega)\hat{v}(\hat{v}-R\omega)}\tilde{v} ,\end{aligned} \tag{40}$$

if the first two terms of a Taylor series expansion of $\ln(\hat{v}-R\omega)$ and $\ln\hat{v}$ are used.

Similarly, the fourth element in $\tilde{\mathbf{U}}$ is simplified to

$$\begin{aligned}\ln\lambda - \ln\hat{\lambda} &= (\ln(v-R\omega) - \ln v) \\ &\quad - (\ln(\hat{v}-R\omega) - ln\hat{v}) = -\frac{R\omega}{v(v-R\omega)}\tilde{v} .\end{aligned} \tag{41}$$

Define

$$K_1 = \max_{\forall i}\{\theta_i\} \tag{42}$$

$$\max\left\{\frac{R\omega_{max}}{v_{min}^2}, \frac{R\omega_{max}}{v_{min}s_{min}}, \frac{R\omega_{max}^2}{v_{min}s_{min}^2}, \frac{R\omega_{max}^2}{v_{min}^2 s_{min}}\right\}$$

where s_{min} is the relative velocity that corresponds to v_{min} and λ_{min}, then $\tilde{\mathbf{U}}\Theta \leq K_1|\tilde{v}|$. In this expression θ_i is the i-th element of Θ and ω_{max} is an upper bound on the angular velocity.