The Application of Constrained Optimal Control to Active Automotive Suspensions

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Abstract

Vehicle suspensions in which forces are generated in response to feedback signals by active elements offer increased design flexibility compared to conventional suspensions using passive elements. Although the design and the synthesis of advanced active suspension can be approached in several different ways, the optimal control techniques seem to constitute the most natural one [5]. Based on the recent result on constrained optimal control theory [2, 3], in this paper we propose a novel optimal controller design where the mechanical constraints of the system components are included explicitly into the controller synthesis. The resulting state feedback control law is continuous and piecewise affine, satisfies the design constraints and can be tuned so that good road holding ability and ride comfort are achieved.

1 Introduction

The interaction between the road and the chassis of an automobile is determined by the suspension system. The wheel suspension most important components are the tire, the spring and the shock absorber mounted between the axle and the chassis. The purpose of the suspension is to adequately support the chassis and to isolate the occupants from the road irregularities while maintaining tire contact with the ground. Good vibration isolation leads to better ride comfort whereas good road holding leads to enhanced safety. In fact, the wheel must have sufficient contact with the ground for the transmission of both lateral and longitudinal forces at any time instant in order to maintain control during manoeuvres. The frictional forces transmitted by the tire are related to the vertical contact force between tire and ground. It therefore follows that the dynamic tire load component must be kept as small as possible. The ability of the suspension system to isolate the chassis from the road surface irregularities can be quantified by considering the vertical acceleration of the vehicle body. The root-mean-square (RMS) value of this acceleration is a measure of the passengers comfort [5].

Suspension systems fall naturally into three categories: passive, semi-active and active systems. Passive suspension systems, which can be found on most conventional cars, use mainly spring and damping elements and require no external power sources to operate. In semi-active systems the dynamic suspension forces are also produced by passive elements such as spring and damper devices, but the parameters of these devices are under control. These parameters may be switched discretely or changed continuously in a slow or rapid fashion. Finally, unlike passive and semi-active systems that can only store or dissipate energy, active suspensions can continually vary the flow of energy and can supply energy to the system when required. Generally, active suspensions are implemented by using an actuator that either replaces or acts in parallel with the passive components.

When designing a suspension system, the dual objective is to minimize the vertical acceleration of the car body and maximize the tire road contact. An important feature of the real world car suspension design problem is that only a fixed and limited working space is available. The suspension working space is defined as the relative distance between axle and vehicle body and is limited by constructional reasons.

Based on what was described above it is clear that optimal control provides an appealing design method for active suspension applications. In fact, the use of optimal control design and synthesis has enjoyed a broad acceptance amongst the active suspension research community, as testified by the references in the survey [5]. Despite its simplicity, linear quadratic optimal control is a method that provides insight into performance potentials and trade-offs, actuator and sensors requirement and optimal system structure. In LQ
design, however, the constraints on the control and state variables, such as the one on the working space, are not considered explicitly. Constraint fulfillment is tested a posteriori with the help of maps depicting the system closed loop behaviour as a function of different tuning parameters of the LQ regulator. The controller design proposed in this paper tries to overcome this limitation by using the recent theory on the solution of constrained optimal control problems developed in [2, 4]. The resulting optimal state feedback control law is continuous and piecewise affine, satisfies input and output constraints and can be tuned so that good road holding ability and ride comfort are achieved. The tuning does not require additional effort since constraints fulfillment is guaranteed for any choice of performance index.

2 Passive Suspension System

2.1 Plant Model

A standard assumption in the design of controllers for active vehicle suspension systems is that the vertical vehicle dynamics can be modeled using four independent quarter-car suspension models. Figure 1(a) shows a typical two degrees of freedom quarter-car model of a passive suspension system.

The body mass \( m \) represents the portion of the sprung mass corresponding to one corner of the vehicle. The sprung mass \( m_{us} \) represents the wheel and axle at one corner. The wheel and axle are connected to the car body through a passive spring-damper combination where \( k_s \) and \( b_s \) are the spring and damping coefficient, respectively. The tire is also modelled as a spring-damper combination where \( k_{ts} \) and \( b_{ts} \) are the spring and damping coefficient, respectively. The variables \( x_s \) and \( x_{us} \) represent the distance to an inertial ground of the sprung mass and unsprung mass, respectively.

The motion equations of this quarter-car model are

\[
\begin{align*}
\dot{x}_s &= b_s \ddot{x}_s + k_s x_3 - b_{ts} \dot{x}_3 - k_{ts} x_3 - b_{ts} \dot{x}_3 - k_{us} x_1 - k_{us} x_1 \\
\dot{x}_{us} &= -k_s x_3 - b_s \dot{x}_3
\end{align*}
\]

(1)

where \( x_1 = x_{us} - r \) is the tire deflection, \( x_2 = \dot{x}_{us} \) is the unsprung mass velocity, \( x_3 = x_s - x_{us} \) is the suspension stroke (also called the working space) and \( x_s = \dot{x}_s \) is the sprung mass velocity. System (1) can be rewritten as

\[
\begin{align*}
\dot{x}(t) &= A^c x(t) + B^c w(t) \\
y(t) &= C x(t)
\end{align*}
\]

(2)

where the input disturbance \( w = \vec{r} \) represents the vertical ground velocity of the road profile and where

\[
A^c = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-\frac{\omega_s^2}{\rho^2} - 2(\zeta_{us} \omega_s + \zeta_s \omega_s) & \omega_s^2 & -\frac{\omega_s \rho}{2 \zeta_{us} \omega_s} & 0 \\
0 & -1 & 0 & 1 \\
0 & 2 \zeta_s \omega_s & -\omega_s^2 & 2(\zeta_{us} \omega_s)
\end{bmatrix}
\]

\[
B^c = \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
1 & 0 & 0 & 0
\end{bmatrix}
\]

The frequencies \( \omega_s \) and \( \omega_{us} \) represent two important parameters in typical automobile suspensions. The frequency \( f_{us} = \frac{\omega_{us}}{2 \pi} \) corresponds to a wheel-hop mode which usually lies in the 8-12 Hz range. The main suspension natural frequency \( f_s = \frac{\omega_s}{2 \pi} \) corresponds to the principal body mode which usually lies in the 1-2 Hz range.

2.2 Simulations

The passive suspension has only two parameters that can be varied in order to optimize its performance: the sprung mass natural frequency \( f_s \) and the corresponding damping ratio \( \zeta_s \). In this section we simulated the linear time invariant system (2) for different combinations of \( f_s \) and \( \zeta_s \). The input disturbance \( w \) is modeled as Gaussian white noise [5]

\[
\begin{align*}
E[w(t)] &= 0, \\
E[w(t_1)w(t_2)] &= 2 \pi W \delta(t_1 - t_2).
\end{align*}
\]

(3)

where \( W \) is equal to the product of the road roughness factor \( A \) and vehicle forward velocity \( V \).

Consider a vehicle with the following parameters

\[
\rho = \frac{m_s}{m_{us}} = 10, \quad \omega_{us} = 2 \pi \cdot 10 \ [\text{rad/s}], \quad \zeta_{us} = 0.
\]

(4)

Since the damping due to the tire is small, the parameter \( \zeta_{us} \) is neglected in the simulations and the suspension damping is determined solely by the damping ratio \( \zeta_s \). We simulated the car (4) driving at constant speed \( V = 25 \text{m/s} \) on a medium quality runway \( (A = 5 \cdot 10^{-6} \text{m}) \) with different combinations of \( f_s \) and \( \zeta_s \). We considered four different natural frequencies
As shown in Figure 1(b) the active suspension model is obtained by replacing the passive components between the wheel and chassis by an actuator.

The motion equations of this quarter-car model are

\[
\begin{align*}
    m_s \ddot{x}_s &= m_s u - b_{us} \dot{x}_1 - k_{us} x_1 \\
    m_s \ddot{x}_4 &= -m_s u
\end{align*}
\]  

(6)

where the input disturbance \( w \) represents the vertical ground velocity of the road profile and \( u \) the actuator vertical acceleration.

The state-space representation of (6) in terms of natural frequencies \( \omega \) and damping factors \( \zeta \) is given by

\[
\begin{align*}
    \dot{x}(t) &= A^c x(t) + B^c u(t) + B^w w(t) \\
    y(t) &= C x(t)
\end{align*}
\]

(7)

with

\[
A^c = \begin{bmatrix}
    0 & -1 & 0 & 0 \\
    -\omega_s^2 & -2\zeta_s \omega_s & 0 & 0 \\
    0 & -1 & 0 & 0 \\
    0 & 0 & -1 & 0
\end{bmatrix},
\]

\[
B^c = \begin{bmatrix}
    0 \\
    \rho \\
    -1
\end{bmatrix},
\]

\[
B^w = \begin{bmatrix}
    -1 \\
    2\zeta_s \omega_s \\
    0
\end{bmatrix},
\]

\[
C = I_{4 \times 4}
\]

This model of a quarter-car active suspension system is unrealistic but nevertheless instructive. Because in practice there are no ideal force generators, a realistic model of a fully active system would be more complex. Our assumption that the four state variables are perfectly measurable is also unrealistic. The tire deflection \( x_1 \) and the absolute velocity \( x_2 \) are not easily measurable although the suspension stroke \( x_3 \) and sprung mass vertical velocity \( x_4 \) could be sensed fairly easily. If we assume that only \( x_3 \) and \( x_4 \) are measured, i.e. \( C = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \), then the pair \((A^c, C)\) defines an observable system and a Kalman filter can be used to estimate the unmeasured state variables.

4 The Optimal Control Problem

The continuous time model (7) is discretized by exact sampling (sampling period \( T_s = 10 \) ms) to obtain the discrete time system

\[
\begin{align*}
    x_{k+1} &= A x_k + B u_k + B w w_k \\
    y_k &= C x_k
\end{align*}
\]

(8)

The control objective is to regulate system (8) to the origin while satisfying constraints on input and states. Moreover the controller should be tuned so that good road holding ability and ride comfort are achieved. To
this aim we define the following infinite horizon con- 
strained optimal control problem

\[
\min_{\{u_k^0, u_k^1, \ldots\}} \sum_{k=0}^{+\infty} x_k^T Q x_k + u_k^T R u_k
\]

(9)

subject to

\[
\begin{align*}
  u_{\min} & \leq u_k & \leq u_{\max}, & k \geq 0 \\
  y_{\min} & \leq y_k & \leq y_{\max}, & k \geq 0 \\
  x_{k+1} & = A x_k + B u_k, & k \geq 0 \\
  y_k & = C x_k, & k \geq 0
\end{align*}
\]

(10)–(13)

where we assume that \( Q = Q' \geq 0, R = R' > 0 \). Note that in the optimization problem (9)–(13) the ground input velocity \( w \) has been neglected. From now on, we will consider the input \( w \) as an unmeasured disturbance and we will not explicitly include it in the controller design.

**Cost Function:** In (9) we set \( Q = diag(r_1, 0, r_2, 0) \) and \( R = 1 \), such that the weighting parameter \( r_1 \) penalizes the tire deflection \( x_1 \) which is a measure of road holding ability while the weighting parameter \( r_2 \) penalizes the suspension stroke \( x_3 \) which is a measure of design constraint. A different tuning of the weights \( r_1 \) and \( r_2 \) will correspond to a different tradeoff between acceleration, suspension travel and tire deflection.

**Constraints:** Input constraints (10) are a function of power and bandwidth of the active device, i.e., the actuator is not capable of delivering any force infinitely fast. More details about the actuator limitations can be found in the survey [5]. In this paper input constraints will not be considered and we will focus on the state constraints (11).

We will distinguish between two types of state constraints: the mechanical constraints and constraints dictated by performance requirements. The first type represents the physical limits of the suspension system, that is the suspension working space \( x_3 \) and the mechanical limits on the tire deflection \( x_1 \) (note that the limits on the tire deflection \( x_1 \) are reached only if the tire is not properly inflated). These constraints will be classified as **hard** constraints.

The second type of state constraints does not represent physical limit but it is introduced in order to fulfill some performance requirements. These constraints can be violated occasionally without leading to some undesirable consequences for the control system. Therefore they are classified as **soft** constraints. For example, a soft constraint can be included on tire deflection \( x_1 \) to guarantee a certain road holding ability whose degradation corresponds to driving in an unsafe state.

We will use the following constraints on the states

\[
\begin{align*}
  \text{hard} : & -3 \leq x_1 \leq 3 \text{ cm}, \ -8 \leq x_3 \leq 8 \text{ cm} \\
  \text{soft} : & -2 \leq x_1 \leq 2 \text{ cm}, \ -6 \leq x_3 \leq 6 \text{ cm}
\end{align*}
\]

(14)

The soft state constraints are included in the problem (9)–(13) by using a slack variable \( \epsilon \) in the equation \( y_{\min} + \epsilon \leq y_k \leq y_{\max} + \epsilon \), and the slack variable \( \epsilon \) is kept small by adding a corresponding penalty term \( \rho_\epsilon \) in the cost function.

**Remark 1** Note that since the ground velocity \( w \) is considered as a disturbance, the solution of (9)–(13) will not guarantee constraint fulfillment for any road profile. For this reason softening the constraints will also ensure the existence of a solution to problem (9)–(13) even for a car driving over a very rough road. One could define a min-max optimal control problem by including the road profile \( w \) directly into the constrained optimal control problem. The resulting solution will be robust with respect to any road input profile \( w \). For more details refer to [3].

In the following we will first consider the standard Linear Quadratic Regulator (LQR) design, obtained by removing constraints (10)–(11) from the optimal control problem. In this case constraint fulfillment can only be verified a posteriori. We will show that in general it is very difficult to tune our LQR to achieve satisfactory performance and constraints fulfillment. Then, we will solve the constrained Linear Quadratic Regulator (CLQR). The resulting optimal state feedback control law is continuous and piecewise affine, satisfies input and output constraints and can be tuned so that good road holding ability and ride comfort are achieved. The tuning does not require any additional effort since constraint fulfillment is guaranteed for any choice of the performance index.

### 4.1 The unconstrained LQR

In this section we describe the main results about the design of an unconstrained LQR-based active suspension published by Hrovat in [5] and [7].

We simulated the vehicle (4) traversing the same medium quality road as in Section 2.2 at a speed of \( V = 25 \text{ m/s} \) and controlled by a LQR. We repeated the simulation 1600 times by varying the weights \( r_1 \) and \( r_2 \) of the matrix \( Q \) (and therefore the corresponding optimal controller). In Figure 3 we show the performance maps parameterized in terms of the weights \( r_1 \) and \( r_2 \). The dashed line represents the passive suspension system of sprung mass natural frequency \( f_s = 1 \text{ Hz} \) and damping ratio \( \zeta_s \) ranging from nearly zero to one. When the car is traversing the same road but at higher speed, or a rougher road at the same speed, the whole performance map moves to the upper right hand corner of the graph. For better quality road and/or low vehicle speed the map tends to move toward the origin.

Once the maps depicted in Figure 3 have been constructed, a tuning leading to good performance and constraints fulfillment can be chosen as explained in [5]. The plots in Figure 3 represent the rms value of the states and it may be confusing if we want to determine
if the constraints are fulfilled. Thus, constraints (14) on the state $x_1$ and $x_3$ are translated into constraints on their rms value. If $x_{1\text{rms}} = m$ then $x < 3 \cdot m \ 99.7\%$ of the time. Suppose, for example, to keep the tire deflection from equilibrium less than $2.54\text{cm}$ for $99.7\%$ of the time, i.e. $x_{1\text{rms}} < \frac{2.54}{3} = 0.847\text{cm}$. From the lower plot of Figure 3 we choose the point $L_2$ corresponding to the minimum rms acceleration possible $(0.3\text{m/s}^2$, i.e. $3\%$ g) and $x_{1\text{rms}} < 0.847\text{cm}$. This point correspond to the choice $r_1 = 1100$ and $r_2 = 100$ ($KLQR_2 = [-16.245, 0.514, -9.521, -4.273]$). Then, from the upper plot of Figure 3, we validate if the chosen tuning satisfies the constraints on the suspension stroke. In our case the values for $r_1$ and $r_2$, depicted as design $L_2$ in the upper plot of Figure 3 correspond to an rms suspension stroke of $x_{3\text{rms}} = 1.58\text{cm}$ which insures that the suspension deflection will remain within $\pm 4.8\text{cm}$.

If our car equipped with the active suspension design $L_2$ is traversing a bad quality road with a road roughness factor $A = 2.5 \cdot 10^{-5}\text{m}$, the resulting rms suspension stroke is $x_{3\text{rms}} = 3.34\text{cm}$ which means that the suspension deflection will remain within $\pm 10\text{cm} \ 99.7\%$ of the time and therefore the suspension will hit its hard limits (14). Thus, we have to consider a more conservative design where the suspension deflection is more penalized. If one considers the suspension design $L_1$ in Figure 3, corresponding to $r_1 = 5000$ and $r_2 = 400$, ($KLQR_1 = [-27.265, 1.062, -18.318, -5.840]$) the resulting rms suspension stroke for the same bad quality road mentioned above, is $x_{3\text{rms}} = 2.65\text{cm}$ which ensures that the suspension travel will remain in the range $\pm 7.9\text{cm} \ 99.7\%$ of time and avoids hitting the mechanical limits.

To conclude, the analysis of these two design cases shows that tuning a controller can be a tough job when constraints on outputs are present. In fact, the weighting parameters have to be chosen to guarantee a certain performance such that the constraints are fulfilled. Thus, the design of an active suspension via the standard optimal control theory may require many trials before a satisfactory control law is found.

![Figure 3](image1.png)  
**Figure 3:** Performance maps of the unconstrained LQR active suspension system parameterized in terms of the weights $r_1$ and $r_2$

4.2 The constrained LQR

Consider the infinite time constrained linear quadratic regulator (CLQR) problem (9)–(13) and its state feedback solution $u^*_k = f_{CLQR}(x_k)$. On a compact set of initial conditions $X^0$, the solution to the CLQR problem (9)–(13) is also continuous and piecewise affine, i.e. $u^*_k = f_{CLQR}(x_k)$ where

$$f_{CLQR}(x_k) = F^i x_k + g^i \text{ if } H^i x \leq K^i, \ i = 1, \ldots, N^r$$

In the optimal control law (15), the number of polyhedral regions $N^r$ depends on the choice of the weights $Q$ and $R$.

In the following we will we solve problem (9)–(13) in the realistic region of operation $X^0 = \left\{ x \in \mathbb{R}^4 \bigg| \begin{bmatrix} -0.3 & -0.2 \\ -0.2 & 0.2 \end{bmatrix} \leq x \leq \begin{bmatrix} 0.3 \\ 0.2 \end{bmatrix} \right\}$ subject to the soft constraints (14) where the slack variable $\epsilon$ is penalized by setting $\rho_\epsilon = 3 \cdot \max\{Q, R\}$. More details of the computation of the control law (15) can be found in [2, 4].

We simulated vehicle (4) traversing the same medium quality road as in Section 2.2 at a speed of $V = 25 \text{ m/s}$ and controlled by the constrained optimal controller (15). We repeated the simulation 100 times.

![Figure 4](image2.png)  
**Figure 4:** Response of the LQR active suspension to a 5 cm bump, upper plot: sprung mass acceleration, center plot: suspension stroke, lower plot: tire deflection
by varying weights $r_1$ and $r_2$ of the matrix $Q$ (and therefore the corresponding piecewise linear optimal controller), and we represented in Figure 5 the performance maps parameterized in terms of the weights $r_1$ and $r_2$. In Figure 5 the dotted lines represent the unconstrained control problem while the vertical dashed line represents the soft constraints (14) on the rms values of the states $x_1$ and $x_3$.

With a fast glance at Figure 5, one realizes that hard constraints are always fulfilled for any choice of the tuning parameters, while soft constraints fulfillment is a function of the tuning. Moreover, we can observe that the tuning of $r_1$ and $r_2$ affects the performance tradeoffs in a similar fashion as in the unconstrained case only until constraints are encountered. In this case the choice of $r_1$ and $r_2$ loses its importance and constraint fulfillment is ensured.

Let now reconsider the design case $L_2$ of Section 4.1 where $r_1 = 1100$ and $r_2 = 100$. In the unconstrained case and for a medium quality road these parameters corresponded to an rms tire deflection $x_{1,rms} = 0.85$cm, an rms suspension stroke $x_{3,rms} = 1.55$cm, and an rms sprung mass acceleration $\dot{x}_{4,rms} = 0.31$m/s$^2$. While the suspension travel constraint is satisfied ($x_3 < 4.7$cm 99.7%(time)), the tire deflection does not fulfill the soft constraints requirement ($x_1 < 2.5$cm 99.7%(time)). The constrained optimal solution $M_2$, corresponding to the same choice of weighting matrices, guarantees hard and soft constraints fulfillment as shown on Figure 5. Therefore, by constraining the tire deflection the road holding ability of the vehicle is improved and the ride comfort is inevitably affected.

In Section 4.1, we showed that the design case $L_2$ was not suitable for driving on a bad quality road because of the too large suspension deflections. Here we consider the same scenario but applied to the controller $M_2$. The rms value of the suspension stroke is $x_{3,rms} = 2.11$cm which means that the suspension travel will remain within $\pm 6.3$cm. Therefore we don’t have to consider a more conservative design, the constrained optimal controller $M_2$ will prevent the suspension to hit the mechanical limit when traversing a very rough road. More simulations are presented in [1].

5 Conclusions

We have shown how the active suspension design fits naturally into the constrained optimal control setting. Based on the recent result on constrained optimal control theory [2, 4, 3], we designed and synthesized a state feedback control law that is continuous and piecewise affine, satisfies the design constraints and that achieves good road holding ability and ride comfort. Several simulation have shown the superior efficacy of the constrained optimal design with respect to the standard LQ design.

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References