Distributed LQR Design for Identical Dynamically Decoupled Systems
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Abstract—We consider a set of identical decoupled dynamical systems and a control problem where the performance index couples the behavior of the systems. The coupling is described through a communication graph where each system is a node and the control action at each node is only function of its state and the states of its neighbors. A distributed control design method is presented which requires the solution of a single LQR problem. The size of the LQR problem is equal to the maximum vertex degree of the communication graph plus one. The design procedure proposed in this paper illustrates how stability of the large-scale system is related to the robustness of local controllers and the spectrum of a matrix representing the desired sparsity pattern of the distributed controller design problem.

I. INTRODUCTION

DISTRIBUTED control techniques today can be found in a broad spectrum of applications ranging from robotics and formation flight to civil engineering. Contributions and interest in this field date back to the early results of [1]. Approaches to distributed control design differ from each other in the assumptions they make on: (i) the kind of interaction between different systems or different components of the same system (dynamics, constraints, objective), (ii) the model of the system (linear, nonlinear, constrained, continuous-time, discrete-time), (iii) the model of information exchange between the systems, (iv) the control design technique used.

In this paper we focus on identical decoupled linear time-invariant systems. Our interest in distributed control for such systems arises from the abundance of networks of independently actuated systems and the necessity of avoiding centralized design when this becomes computationally prohibitive. Networks of vehicles in formation, production units in a power plant, cameras at an airport, an array of mechanical actuators for deforming a surface are just a few examples.

In a descriptive way, the problem of distributed control for decoupled systems can be formulated as follows. A dynamical system is composed of (or can be decomposed into) distinct dynamical subsystems that can be independently actuated. The subsystems are dynamically decoupled but have common objectives, which make them interact with each other. Typically the interaction is local, i.e., the goal of a subsystem is a function of only a subset of other subsystems’ states. The interaction will be represented by an interaction graph, where the nodes represent the subsystems and an edge between two nodes denotes a coupling term in the controller associated with the nodes. Also, typically it is assumed that the exchange of information has a special structure, i.e., it is assumed that each subsystem can sense and/or exchange information with only a subset of other subsystems. We will assume that the interaction graph and the information exchange graph coincide. A distributed control scheme consists of distinct controllers, one for each subsystem, where the inputs to each subsystem are computed only based on local information, i.e., on the states of the subsystem and its neighbors.

Over the past few years, there has been a renewal of interest in systems composed of a large number of interacting and cooperating interconnected units [2]–[20]. Recent advances in stability analysis of such systems yielded greater insight into the relationship between the spectrum of the interconnection graph and global stability of the overall system. Research efforts in this area are epitomized by the pioneering work of [5], which provided stability analysis tools for an interconnection of identical linear dynamical systems each using the same control law that operates on the average information obtained from neighbors.

Most recent results, however, pertain to the study of distributed parameter systems where the underlying dynamics are spatially invariant, and where the controls and measurements are spatially distributed. The fundamental work of [4], [6] in this field discusses distributed LQR design purely for infinite dimensional spatially invariant systems, where the problem diagonalizes exactly into a parameterized family of finite dimensional LQR problems. It is established that the corresponding ARE solutions are translation-invariant operators, and the optimal controller is a spatially invariant system. The authors show that quadratically optimal controllers for spatially invariant systems are themselves spatially invariant. Another recent result for spatially invariant systems and unbounded domains in [7] considers the problem of distributed controller design, when communication among the sites is limited. In particular, the controller is assumed to be constrained so that information is propagated with a delay that depends on the distance between subsystems. The authors show that the problem of optimal design can be cast as a convex problem provided that the propagation speeds in the controller are at least as fast as those in the plant. Special spatially invariant structures composed of an infinite string of linear systems have also received much attention. The paper [8] analyzes inverse optimality of localized distributed controllers in such systems.

Another significant group of recently explored strategies involves relaxations to the LMI versions of these spatially
invariant problems. In [9], authors consider heterogeneous subsystems and a relaxation, which is used to derive sufficient conditions for the existence of controllers which stabilize the system and provide a guaranteed level of performance. The work in [10] is concerned with finding distributed controllers for a set of arbitrarily connected, finite and possibly heterogeneous LTI systems. The authors consider nonoriented edges in the graph interconnection and formulate convex conditions for the existence of output-feedback controllers, which achieve a certain $H_{\infty}$ performance. The resulting linear matrix inequalities grow in size with the number of systems in the interconnection.

The complexity associated with the computation of distributed optimal controllers is well exemplified by the study in [11]. There, a finite-time LQR synthesis problem in discrete time is considered, where the matrix describing the control law is constrained to lie in a particular vector space. This vector space represents a pre-specified distributed control structure between a network of autonomous agents. The computationally intensive optimal solution for the distributed control problem is presented along with a computationally more tractable sub-optimal one.

One of the most notable recent results in characterizing the complexity of optimal distributed controller design subject to constraints on the controller structure was presented in [12]. The authors establish that in general, efficient solution of such problems require a special property of the sparsity pattern or interconnection structure to hold. Specifically, it is shown that quadratic invariance of the assumed controller structure implies that the distributed minimum-norm problem may be solved efficiently via convex programming.

This manuscript proposes a simple distributed controller design approach and focuses on a class of systems, for which existing methods outlined above are either not efficient or would not even be directly applicable. Our method applies to large-scale systems composed of finite number of identical subsystems where the interconnection structure or sparsity pattern is not required to have any special invariance properties. The philosophy of our approach builds on the recent works [13]–[15], where at each node, the model of its neighbors are used to predict their behavior. We show that in absence of state and input constraints, and for identical linear system dynamics, such an approach leads to an extremely powerful result: the synthesis of stabilizing distributed control laws can be obtained by using a simple local LQR design, whose size is limited by the maximum vertex degree of the interconnection graph plus one. Furthermore, the design procedure proposed in this paper illustrates how stability of the overall large-scale system is related to the robustness of local controllers and the spectrum of a matrix representing the desired sparsity pattern. In addition, the constructed distributed controller is stabilizing independent of the tuning parameters in the local LQR cost function. This leads to a method for designing distributed controllers for a finite number of dynamically decoupled systems, where the local tuning parameters can be chosen to obtain a desirable global performance. Such result can be immediately used to improve current stability analysis and controller synthesis in the field of distributed receding horizon control for dynamically decoupled systems [3], [13], [16]–[18]. We emphasize that the aforementioned advantages are the results of two main simplifying assumptions compared to a general structured optimal design: identical and decoupled subsystem dynamics and suboptimality of the global controller.

The paper is organized as follows. In Section II, we study the solution properties of the LQR for a set of identical, decoupled linear systems. Such properties will be used in the paper to construct a stabilizing distributed controller for arbitrary interconnection structures. Section III summarizes important properties of graph Laplacian and adjacency matrices, which will be useful in our proofs. Section IV presents the stabilizing distributed controller design procedure using the local LQR solution properties. The proposed distributed control design method and the effect of the free tuning parameters in the local LQR design is illustrated by a simulation example in Section V. Some concluding remarks are made in Section VI.

**Notation and Preliminaries**

We denote by $\mathbb{R}$ the field of real numbers, $\mathbb{C}$ the field of complex numbers and $\mathbb{R}^{m \times n}$ the set of $m \times n$ real matrices.

$$\mathbb{C}^{-} = \{ s \in \mathbb{C} : \text{Re}(s) < 0 \}, \quad \mathbb{C} = \{ s \in \mathbb{C} : \text{Re}(s) \leq 0 \}$$

The transpose of a vector $x$ and a matrix $M$ will be denoted by $x'$ and $M'$, respectively. A matrix $M \in \mathbb{R}^{n \times n}$ is symmetric if $M = M'$.

**Notation 1:** Let $M \in \mathbb{R}^{m \times n}$, then $M[i : j, k : l]$ denotes a matrix of dimension $(j - i + 1) \times (l - k + 1)$ obtained by extracting rows $i$ to $j$ and columns $k$ to $l$ from the matrix $M$, with $m \geq j \geq i \geq 1, n \geq k \geq l \geq 1$.

**Notation 2:** $I_m$ denotes the identity matrix of dimension $m$, $I_m \in \mathbb{R}^{m \times m}$.

**Notation 3:** Let $\lambda_i(M)$ denote the $i$-th eigenvalue of $M \in \mathbb{R}^{n \times n}, i = 1, \ldots, n$. The spectrum of $M$ will be denoted by $\mathcal{S}(M) = \{ \lambda_1(M), \ldots, \lambda_n(M) \}$.

**Definition 1:** Let $A, B \in \mathbb{R}^{n \times n}$ be symmetric.

1) $A$ is positive definite if $x'Ax > 0$ for all nonzero $x \in \mathbb{R}^n$, and $A$ is positive semidefinite if $x'Ax \geq 0$ for all nonzero $x \in \mathbb{R}^n$. We denote this by $A > 0$ and $A \geq 0$, respectively.

2) $A$ is negative (semi) definite if $-A$ is positive (semi) definite.

3) $A < B$ and $A \leq B$ mean $A - B < 0$ and $A - B \leq 0$, respectively.

**Definition 2:** A matrix $M \in \mathbb{R}^{n \times n}$ is called Hurwitz (or stable) if all its eigenvalues have negative real part, i.e. $\lambda_i(M) \in \mathbb{C}^{-}, i = 1, \ldots, n$.

**Notation 4:** Let $A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{p \times q}$. Then $A \otimes B$ denotes the Kronecker product of $A$ and $B$:

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} \in \mathbb{R}^{mp \times nq}. \quad (1)$$

**Proposition 1:** Consider two matrices $A = \alpha I_n$ and $B \in \mathbb{R}^{n \times n}$. Then $\lambda_i(A + B) = \alpha + \lambda_i(B), \ i = 1, \ldots, n$.

**Proof:** Take any eigenvalue $\lambda_i(B)$ and the corresponding eigenvector $v_i \in \mathbb{C}^n$. Then $(A + B)v_i = Av_i + Bv_i = \alpha v_i + \lambda_i(B)v_i = (\alpha + \lambda_i(B))v_i \in \mathbb{C}^n$. Therefore, $\lambda_i(A + B) = \alpha + \lambda_i(B)$.
Proposition 2: Given $A, C \in \mathbb{R}^{m \times m}$ and $B \in \mathbb{R}^{m \times n}$, consider two matrices $\hat{A} = I_n \otimes A$ and $\bar{C} = B \otimes C$, where $\hat{A}, \bar{C} \in \mathbb{R}^{mN \times mN}$. Then $S(\hat{A} + \bar{C}) = \bigcup_{i=1}^{nL} S(A + \lambda_i(B)C)$, where $\lambda_i(B)$ is the $i$-th eigenvalue of $B$.

Proof: Let $v \in \mathbb{C}^m$ be an eigenvector of $B$ corresponding to $\lambda(B)$, and $u \in \mathbb{C}^m$ be an eigenvector of $M = (A + \lambda(B)C)$ with $\lambda(M)$ as the associated eigenvalue. Consider the vector $v \otimes u \in \mathbb{C}^{mn}$. Then $(\hat{A} + \bar{C})(v \otimes u) = v \otimes Au + Bu \otimes Cu = v \otimes Au + \lambda(B)v \otimes Cu = v \otimes (Au + \lambda(B)Cu)$. Since $(A + \lambda(B)C)u = \lambda(M)u$, we get $(\hat{A} + \bar{C})(v \otimes u) = \lambda(M)(v \otimes u)$.

II. LQR Properties for Dynamically Decoupled Systems

Consider a set of $N_L$ identical, decoupled linear time-invariant dynamical systems, the $i$-th system being described by the continuous-time state equation:

$$
\dot{x}_i = Ax_i + Bu_i,
$$

where $x_i(t) \in \mathbb{R}^n$, $u_i(t) \in \mathbb{R}^m$ are states and inputs of the $i$-th system at time $t$, respectively. Let $\bar{x}(t) \in \mathbb{R}^{nN_L}$ and $\bar{u}(t) \in \mathbb{R}^{mN_L}$ be the vectors which collect the states and inputs of the $N_L$ systems at time $t$:

$$
\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}\bar{u},
$$

$$
\bar{x}(0) = x_{0i} \triangleq [x_{1i,0}, \ldots, x_{N_L,0}],
$$

with

$$
\bar{A} = I_{N_L} \otimes A, \quad \bar{B} = I_{N_L} \otimes B.
$$

We consider an LQR control problem for the set of $N_L$ systems where the cost function couples the dynamic behavior of individual systems:

$$
J(\bar{u}, \bar{x}_0) = \int_0^\infty \left( \sum_{i=1}^{N_L} (x_i(\tau)'Q_{ii}x_i(\tau) + u_i(\tau)'R_{ii}u_i(\tau)) + \sum_{i=1}^{N_L} \sum_{j \neq i} (x_i(\tau) - x_j(\tau))'Q_{ij}(x_i(\tau) - x_j(\tau)) \right) d\tau
$$

with

$$
R_{ii} = R_{ii}', R > 0, \quad Q_{ii} = Q_{ii}', Q_{ij} \geq 0 \quad \forall i, \quad Q_{ij} = Q_{ji} \geq 0 \quad \forall i \neq j.
$$

The cost function (5) contains terms which weigh the $i$-th system states and inputs, as well as the difference between the $i$-th and the $j$-th system states and can be rewritten using the following compact notation:

$$
J(\bar{u}(t), \bar{x}_0) = \int_0^\infty (\bar{x}(\tau)'\bar{Q}\bar{x}(\tau) + \bar{u}(\tau)\bar{R}\bar{u}(\tau)) \, d\tau,
$$

where the matrices $\bar{Q}$ and $\bar{R}$ have a special structure defined next. $\bar{Q}$ and $\bar{R}$ can be decomposed into $N^2_L$ blocks of dimension $n \times n$ and $m \times m$ respectively:

$$
\bar{Q} = \begin{bmatrix} \bar{Q}_{11} & \cdots & \bar{Q}_{1N_L} \\ \vdots & \ddots & \vdots \\ \bar{Q}_{N_L1} & \cdots & \bar{Q}_{N_LN_L} \end{bmatrix}, \quad \bar{R} = \begin{bmatrix} R & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & R \end{bmatrix}
$$

with

$$
\bar{Q}_{ii} = Q_{ii} + \sum_{k=1, k \neq i}^{N_L} Q_{ik}, \quad i = 1, \ldots, N_L.
$$

$$
\bar{R} = \bar{Q}_{i,j} = -Q_{ij}, \quad i, j = 1, \ldots, N_L, \quad i \neq j.
$$

Remark 1: The cost function structure (5) can be used to describe several practical applications including formation flight, paper machine control and monitoring networks of cameras [19], [21].

Let $\tilde{K}$ and $\tilde{z}_0'\tilde{P}\tilde{x}_0$ be the optimal controller and the value function corresponding to the following LQR problem:

$$
\min_{\bar{u}} J(\bar{u}, \bar{x}_0)
$$

subject to

$$
\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}\bar{u},
$$

$$
\bar{x}(0) = \bar{x}_0
$$

Throughout the paper we will assume that a stabilizing solution to the LQR problem (10) with finite performance index exists and is unique (see [22], p. 52 and references therein):

Assumption 1: System $\bar{A}, \bar{B}$ is stabilizable and system $\bar{A}, \bar{C}$ is observable, where $\bar{C}$ is any matrix such that $\bar{C}'\bar{C} = \bar{Q}$.

We will also assume local stabilizability and observability:

Assumption 2: System $A, B$ is stabilizable and systems $A, C$ are observable, where $C$ is any matrix such that $C'C = Q$.

It is well known that

$$
\tilde{K} = -\tilde{R}^{-1}\tilde{B}'\tilde{P},
$$

where $\tilde{P}$ is the symmetric positive definite solution to the following ARE:

$$
\tilde{A}'\tilde{P} + \tilde{P}\tilde{A} - \tilde{P}\tilde{B}\tilde{R}^{-1}\tilde{B}'\tilde{P} + \bar{Q} = 0
$$

We decompose $\tilde{K}$ and $\tilde{P}$ into $N^2_L$ blocks of dimension $m \times n$ and $n \times n$, respectively. Denote by $\tilde{K}_{ij}$ and $\tilde{P}_{ij}$ the $(i, j)$ block of the matrix $\tilde{K}$ and $\tilde{P}$, respectively. In the following theorems we show that $\tilde{K}_{ij}$ and $\tilde{P}_{ij}$ satisfy certain properties which will be critical for the design of stabilizing distributed controllers in Section IV. These properties stem from the special structure of the LQR problem (10). Next, the matrix $X$ is defined as $X = BR^{-1}B'$.
Then, in equation (10),

\[
N = 1 + \tilde{N}_l = \sum_{j=1}^{N_L} \tilde{P}_{ij} - \sum_{j=1}^{N_L} \tilde{P}_{ji} \]

for a certain \( \tilde{F}_{ii} \). In general, \( \tilde{F}_{ii} \) would be a function of \( \sum_{j=1}^{N_L} \tilde{P}_{ij} \). We will show next that for the LQR problem (10), \( \tilde{F}_{ii} = P \) and that it is not a function of \( \sum_{j=1}^{N_L} \tilde{P}_{ij} \).

The equations for the diagonal blocks \( \tilde{P}_{ii} \) of \( \tilde{P} \) in the ARE equation (12) are

\[
A' \tilde{P}_{ii} + \tilde{P}_{ii} A - \left( \sum_{k=1}^{N_L} \tilde{P}_{ik} \right) = 0
\]

(15)

for \( i = 1, \ldots, N_L \). Note also that \( \tilde{P}_{ij} = \tilde{P}_{ji} \). Substituting (14) in the above equation leads to

\[
A' \tilde{F}_{ii} + \tilde{F}_{ii} A - A' \left( \sum_{k=1}^{N_L} \tilde{P}_{ik} \right) A - \left( \sum_{k=1}^{N_L} \tilde{P}_{ik} \right) = 0
\]

(16)

\[
+ \left( \sum_{k=1}^{N_L} \tilde{P}_{ik} \right) X \left( \sum_{l=1}^{N_L} \tilde{P}_{il} \right) + \sum_{k=1}^{N_L} \tilde{P}_{ik} X \tilde{F}_{ii} + \tilde{F}_{ii} X \left( \sum_{k=1}^{N_L} \tilde{P}_{ik} \right) + \tilde{Q}_{ii} = 0.
\]

The equations for the off-diagonal blocks \( \tilde{P}_{ij}, i \neq j \) of \( \tilde{P} \) in the ARE equation (12) are

\[
A' \tilde{P}_{ij} + \tilde{P}_{ij} A - \left( \sum_{k=1}^{N_L} \tilde{P}_{ik} \right) \tilde{P}_{kj} = 0
\]

(17)

for \( i, j = 1, \ldots, N_L, i \neq j \). Substituting (14) in the above equation leads to

\[
A' \tilde{F}_{ij} + \tilde{F}_{ij} A - \left( \sum_{k=1}^{N_L} \tilde{P}_{ik} \right) \tilde{P}_{kj} \]

\[
+ \tilde{P}_{ij} X \left( \sum_{k=1}^{N_L} \tilde{P}_{jk} \right) - \left( \sum_{k=1}^{N_L} \tilde{P}_{ik} \right) \tilde{Q}_{ij} = 0.
\]

(18)

Summing up equation (18) for all \( j \neq i \) corresponds to a block-wise row sum of off-diagonal terms in the \( i \)-th block row of the ARE equation (12). This summation leads to

\[
A' \left( \sum_{k=1}^{N_L} \tilde{P}_{ik} \right) + \left( \sum_{k=1}^{N_L} \tilde{P}_{ik} \right) A - \left( \sum_{k=1}^{N_L} \tilde{P}_{ik} \right) \tilde{F}_{ii} X \tilde{P}_{il} + \left( \sum_{k=1}^{N_L} \tilde{P}_{ik} \right) X \tilde{P}_{il} \]

\[
+ \sum_{l=1}^{N_L} \tilde{P}_{lk} \frac{X}{\tilde{P}_{il}} \left( \sum_{k=1}^{N_L} \tilde{P}_{ik} \right) - \sum_{l=1}^{N_L} \tilde{P}_{lk} \frac{X}{\tilde{P}_{il}} \left( \sum_{k=1}^{N_L} \tilde{P}_{ik} \right) \tilde{F}_{ii} X \tilde{P}_{il} \frac{X}{\tilde{P}_{il}} - \sum_{l=1}^{N_L} \tilde{P}_{lk} X \tilde{F}_{kl} + \sum_{k=1}^{N_L} \tilde{Q}_{ik} = 0.
\]

(19a)

Notice that

\[
\sum_{k=1}^{N_L} \tilde{P}_{ik} \tilde{F}_{kl} - \sum_{l=1}^{N_L} \tilde{P}_{lk} \tilde{F}_{kl} = \sum_{k=1}^{N_L} \tilde{P}_{ik} \tilde{F}_{ki},
\]

(20)

where we used symmetry, switching of sum operators and renaming the indices. Adding equation (19) to equation (16) and using properties of \( Q \) defined in (9), equally numbered terms will cancel each other out leading to

\[
A' \tilde{F}_{ii} + \tilde{F}_{ii} A - \left( \sum_{k=1}^{N_L} \tilde{P}_{ik} \right) \tilde{F}_{ii} X \tilde{P}_{il} + \sum_{k=1}^{N_L} \tilde{P}_{ik} X \left( \tilde{F}_{ii} - \tilde{F}_{kk} \right) + \tilde{Q}_{ii} = 0.
\]

(21)

Summing up equation (21) over all \( i = 1, \ldots, N_L \) we obtain

\[
\sum_{i=1}^{N_L} \left( A' \tilde{F}_{ii} + \tilde{F}_{ii} A - \tilde{F}_{ii} X \tilde{F}_{ii} + \tilde{Q}_{ii} \right) = 0,
\]

(22)

which proves the theorem. \( \Box \)

Next we particularize Theorem 1 to the case of identical weights and prove an additional property of LQR for decoupled systems.

**Theorem 2:** Assume the weighting matrices (9) of the LQR problem (10) are chosen as

\[
Q_{ii} = Q_1 \quad \forall i = 1, \ldots, N_L
\]

\[
Q_{ij} = Q_2 \quad \forall i, j = 1, \ldots, N_L, i \neq j.
\]

(23)

Let \( \mathbb{P}_{\tilde{X}} \) be the value function of the LQR problem (10) with weights (23), and the blocks of the matrix \( P \) be denoted by \( \tilde{P}_{ij} = P[(i-1)n : in, (j-1)n : jn] \) with \( i, j = 1, \ldots, N_L \).
The assumption in (23) requires that the weight \( Q_1 \) used for absolute states and the weight \( Q_2 \) used for neighboring state differences are equal for all nodes and for all neighbors of a node, respectively. Such an assumption and the fact that \( \bar{A} \) and \( \bar{B} \) are block-diagonal with identical blocks, imply that the ARE in (12) is a set of \( N_L \) identical equations where the matrices \( \tilde{P}_{ij} \) are all identical and symmetric for all \( i \neq j \). We denote by \( \tilde{P}_2 \), the generic block \( \tilde{P}_{ij} \) for \( i \neq j \). The matrices \( F_{ii} = \sum_{j=1}^{N_L} \tilde{P}_{ij} \) defined in (14) are all identical and therefore equation (15) in Theorem 1 becomes:

\[
N_L \left( A'\tilde{F}_{ii} + \tilde{F}_{ii}A - \tilde{F}_{ii}X\tilde{F}_{ii} + Q_1 \right) = 0,
\]

which proves property (I) with \( F_{ii} = P \). Property (II) follows from property (I) and from equation (11) which implies that \( \tilde{K}_{ij} = -R^{-1}B'\tilde{P}_{ij} \). Next we prove property (III).

The ARE equations (17) for the block \( \tilde{P}_{ij} \) with \( i \neq j \) become

\[
A'\tilde{P}_{2} + \tilde{P}_{2}A - \tilde{P}_{2}X\tilde{P}_{2} - \tilde{P}_{1}X\tilde{P}_{2} - (N_L-2)\tilde{P}_{2}X\tilde{P}_{2} - Q_2 = 0,
\]

which can be rewritten as follows in virtue of property (I):

\[
(A - XP)'\tilde{P}_{2} + \tilde{P}_{2}(A - XP) + (N_L)\tilde{P}_{2}X\tilde{P}_{2} - Q_2 = 0.
\]

where \( P \) is the symmetric positive definite solution of the ARE (24) associated with a single node local problem.

Rewrite equation (27) as

\[
(A - XP)'(-N_L\tilde{P}_{2}) + (-N_L\tilde{P}_{2})(A - XP) - (N_L)\tilde{P}_{2}X(-N_L\tilde{P}_{2}) + N_LQ_2 = 0.
\]

Since \( X > 0 \) and \( Q_2 \geq 0 \), equation (28) can be seen as an ARE associated with an LQR problem for the stable system \((A - XP, B)\) with weights \( N_LQ_2 \) and \( R \). Let the matrix \(-N_L\tilde{P}_{2}\) be its positive semidefinite solution. Then, the following matrix

\[
\hat{P} = \begin{bmatrix}
P - (N_L - 1)\tilde{P}_{2} & \tilde{P}_{2} & \cdots & \tilde{P}_{2} \\
\tilde{P}_{2} & P - (N_L - 1)\tilde{P}_{2} & \cdots & \tilde{P}_{2} \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{P}_{2} & \cdots & \cdots & P - (N_L - 1)\tilde{P}_{2}
\end{bmatrix}
\]

is a symmetric positive definite matrix since

\[
\hat{x}'\hat{P}\hat{x} = x_i'P_{xx} + \sum_{i=1}^{N_L} \sum_{j \neq i} (x_i - x_j)'(\tilde{P}_{2}(x_i - x_j) - \tilde{P}_{1})(x_i - x_j)
\]

and it is the unique symmetric positive definite matrix solution to the ARE (12). This proves the theorem. \( \square \)

Under the hypothesis of Theorem 2, because of symmetry and equal weights \( Q_2 \) on the neighboring state differences and equal weights \( Q_1 \) on absolute states, the LQR optimal controller will have the following structure:

\[
\hat{K} = \begin{bmatrix}
K_1 & K_2 & \cdots & K_2 \\
K_2 & K_1 & \cdots & K_2 \\
\vdots & \vdots & \ddots & \vdots \\
K_2 & \cdots & \cdots & K_1
\end{bmatrix},
\]

with \( K_1 \) and \( K_2 \) functions of \( N_L, A, B, Q_1, Q_2 \) and \( R \).

Remark 2: We conjecture that Theorem 2 is true under much milder assumptions on the tuning parameters (even with different weights on neighboring state differences) as long as the cost function is defined as in (9). The authors are currently studying this conjecture.

The following corollaries of Theorem 2 follow from the stability and the robustness of the LQR controller \(-R^{-1}B'(-N_L\tilde{P}_{2})\) for system \((A - XP, B)\) in (28).

**Corollary 1:** \( A - XP + N_L\tilde{P}_{2} \) is a Hurwitz matrix.

From the gain margin properties [23] we have:

**Corollary 2:** \( A - XP + \alpha N_L\tilde{P}_{2} \) is a Hurwitz matrix for all \( \alpha > \frac{1}{2} \), with \( \alpha \in \mathbb{R} \).

Essentially, Condition 1 characterizes systems for which the LQR gain stability margin described in Corollary 2 is extended to any positive \( \alpha \). The fact that \( A - XP \) is already stable, does not necessarily guarantee this property.

Checking the validity of Condition 1 for a given tuning of \( P \) and \( \tilde{P}_2 \) may be performed as a stability test for a simple affine parameter-dependent model

\[
\dot{x} = (A_0 + \alpha A_1)x,
\]

where \( A_0 = A - XP, A_1 = N_L\tilde{P}_{2} \) and \( 0 \leq \alpha \leq \frac{1}{2} \).

This test can be posed as an LMI problem (Proposition 5.9 in [24]) searching for quadratic parameter-dependent Lyapunov functions.

In the following section we introduce some basic concepts of graph theory before presenting the distributed control design problem.

### III. LAPLACIAN SPECTRUM OF GRAPHS

This section is a concise review of the relationship between the eigenvalues of a Laplacian matrix and the topology of the associated graph. We refer the reader to [25], [26] for a comprehensive treatment of the topic. We list a collection of properties associated with undirected graph Laplacians and adjacency matrices, which will be used in subsequent sections of the paper.
A graph $G$ is defined as
\[ G = (V, A) \tag{33} \]
where $V$ is the set of nodes (or vertices) $V = \{1, \ldots, N\}$ and $A \subseteq V \times V$ the set of edges $(i, j)$ with $i \in V$, $j \in V$. The degree $d_i$ of a vertex graph $j$ is the number of edges which start from $j$. Let $d_{max}(G)$ denote the maximum vertex degree of the graph $G$.

We denote by $A(G)$ the $(0, 1)$ adjacency matrix of the graph $G$. Let $A_{i,j} \in \mathbb{R}$ be its $i,j$ element, then $A_{i,j} = 0$, $\forall i = 1, \ldots, N$, $A_{i,j} = 1$ if $(i, j) \notin A$ and $A_{i,j} = 1$ if $(i, j) \in A$, $\forall i,j = 1, \ldots, N, i \neq j$. We will focus on undirected graphs, for which the adjacency matrix $A(G)$ is symmetric.

Let $S(A(G)) = \{\lambda_1(A(G)), \ldots, \lambda_N(A(G))\}$ be the spectrum of the adjacency matrix $A$ associated with an undirected graph $G$ arranged in nondecreasing order.

**Property 1:** \(\lambda_N(A(G)) \leq d_{max}(G)\).

This property together with Proposition 1 implies

**Property 2:** $\gamma_i \geq 0, \forall \gamma_i \in S(d_{max} I_N - A)$.

We define the Laplacian matrix of a graph $G$ in the following way
\[ L(G) = D(G) - A(G), \tag{34} \]
where $D(G)$ is the diagonal matrix of vertex degrees $d_i$ (also called the valence matrix). Eigenvalues of Laplacian matrices have been widely studied by graph theorists. Their properties are strongly related to the structural properties of their associated graphs. Every Laplacian matrix is a singular matrix. By Gershgorin’s theorem [27], the real part of each nonzero eigenvalue of $L(G)$ is strictly positive.

For undirected graphs, $L(G)$ is a symmetric, positive semidefinite matrix, which has only real eigenvalues. Let $S(L(G)) = \{\lambda_1(L(G)), \ldots, \lambda_N(L(G))\}$ be the spectrum of the Laplacian matrix $L$ associated with an undirected graph $G$ arranged in nondecreasing order. Then,

**Property 3:**
1) $\lambda_1(L(G)) = 0$ with corresponding eigenvector of all ones, and $\lambda_2(L(G)) \neq 0$ iff $G$ is connected. In fact, the multiplicity of 0 as an eigenvalue of $L(G)$ is equal to the number of connected components of $G$.
2) The modulus of $\lambda_i(L(G))$, $i = 1, \ldots, N$ is less then $\sqrt{N}$.

The second smallest Laplacian eigenvalue $\lambda_2(L(G))$ of graphs is probably the most important information contained in the spectrum of a graph. This eigenvalue, called the algebraic connectivity of the graph, is related to several important graph invariants, and it has been extensively investigated.

Let $L(G)$ be the Laplacian of a graph $G$ with $N$ vertices and with maximal vertex degree $d_{max}(G)$. Then properties of $\lambda_2(L(G))$ include

**Property 4:**
1) $\lambda_1(L(G)) \leq \frac{N}{N-2} \min(d(v), v \in V)$,
2) $\lambda_2(L(G)) \leq \nu(G) \leq \eta(G)$,
3) $\lambda_2(L(G)) \geq 2\eta(G)$,
4) $\lambda_2(L(G)) \geq 2(\cos \frac{\pi}{N} - \cos 2\frac{\pi}{N})\eta(G) - 2\cos \frac{\pi}{N} \sin \frac{\pi}{N} d_{max}(G)$,

where $\nu(G)$ is the vertex connectivity of the graph $G$ (the size of a smallest set of vertices whose removal renders $G$ disconnected) and $\eta(G)$ is the edge connectivity of the graph $G$ (the size of a smallest set of edges whose removal renders $G$ disconnected) [28].

Further relationships between the graph topology and Laplacian eigenvalue locations are discussed in [26] for undirected graphs. Spectral characterization of Laplacian matrices for directed graphs can be found in [27].

IV. DISTRIBUTED CONTROL DESIGN

We consider a set of $N$ linear, identical and decoupled dynamical systems, described by the continuous-time time-invariant state equation (2), rewritten below
\[ \dot{x}_i = Ax_i + Bu_i, \]
\[ x_i(0) = x_{i0}, \]
where $x_i(t) \in \mathbb{R}^n$, $u_i(t) \in \mathbb{R}^m$ are states and inputs of the $i$-th system at time $t$, respectively. Let $\hat{x}(t) \in \mathbb{R}^{Nn}$ and $\hat{u}(t) \in \mathbb{R}^{Nm}$ be the vectors which collect the states and inputs of the $N$ systems at time $t$,
\[ \hat{x} = \hat{A}\hat{x} + \hat{B}\hat{u}, \]
\[ \hat{x}(0) = \hat{x}_0 \triangleq [x_{10}, \ldots, x_{N0}]', \tag{35} \]
with
\[ \hat{A} = I_N \otimes A, \]
\[ \hat{B} = I_N \otimes B. \]

**Remark 4:** Systems (35) and (3) differ only in the number of subsystems. We will use system (3) with $N_L$ subsystems when referring to local problems, and system (35) with $N$ subsystems when referring to the global problem. Accordingly, tilded matrices will refer to local problems and hatted matrices will refer to the global problem.

We use a graph to represent the coupling in the control objective and the communication in the following way. We associate the $i$-th system with the $i$-th node of a graph $G = (V, A)$. If an edge $(i, j)$ connecting the $i$-th and $j$-th node is present, then 1) the $i$-th system has full information about the state of the $j$-th system and, 2) the $i$-th system control law minimizes a weighted distance between the $i$-th and the $j$-th system states.

The class of $K^N_{n,m}(G)$ matrices is defined as follows:

**Definition 3:** $K^N_{n,m}(G) = \{M \in \mathbb{R}^{Nn \times mN} | M_{ij} = 0 \text{ if } (i, j) \notin A, M_{ij} = M[(i-1)n : in, (j-1)m : jm], i, j = 1, \ldots, N\}$

The distributed optimal control problem is defined as follows:

\[ \min_{K} J(\hat{u}, \hat{x}_0) = \int_{0}^{\infty} \left( \dot{\hat{x}}(\tau)' \hat{Q} \dot{\hat{x}}(\tau) + \hat{u}(\tau)' \hat{R} \hat{u}(\tau) \right) d\tau, \tag{36a} \]
subject to $\dot{x} = \hat{A}\hat{x} + \hat{B}\hat{u}$, $\hat{u}(t) = \hat{K}\hat{x}(t)$, $\hat{K} \in K^N_{n,m}(G)$, $\hat{Q} \in K^N_{n,m}(G)$, $\hat{R} = I_N \otimes R$, $\hat{x}(0) = \hat{x}_0$ \tag{36f}

with $\hat{Q} = \hat{Q}' \geq 0$ and $\hat{R} = \hat{R}' > 0$. We also refer to problem (36) without (36d) as a centralized optimal control
In general, computing the solution to problem (36) is an NP-hard problem. Next, we propose a suboptimal control design leading to a controller \( \hat{K} \) with the following properties:

1. \( \hat{K} \in K_{m,n}^N(G) \)
2. \( \hat{A} + \hat{B}K \) is Hurwitz.
3. Simple tuning of absolute and relative state errors, and control effort within \( \hat{K} \).

Such controller will be referred to as distributed suboptimal controller. The following theorem will be used to propose a distributed suboptimal control design procedure.

\textbf{Theorem 3:} Consider the LQR problem (10) with \( N_L = d_{\text{max}}(G) + 1 \) and weights chosen as in (23) and its solution (29), (31).

Let \( M \in \mathbb{R}^{N \times N} \) be a symmetric matrix with the following property:

\[
\lambda_i(M) > \frac{N_L}{2}, \quad \forall \lambda_i(M) \in \mathcal{S}(M) \setminus \{0\}. \tag{38}
\]

and construct the feedback controller:

\[
\hat{K} = -I_N \otimes R^{-1}B^TP + M \otimes R^{-1}B^T\hat{P}_2. \tag{39}
\]

Then, the closed loop system

\[
A_{cl} = I_N \otimes A + (I_N \otimes B)\hat{K} \tag{40}
\]

is asymptotically stable.

\textbf{Proof:} Consider the eigenvalues of the closed-loop system \( A_{cl} \):

\[
\mathcal{S}(A_{cl}) = \mathcal{S} \left( I_N \otimes (A - XP) + M \otimes (X\hat{P}_2) \right) \]

By Proposition 2:

\[
\mathcal{S} \left( I_N \otimes (A - XP) + M \otimes (X\hat{P}_2) \right) = \bigcup_{i=1}^{N} \mathcal{S} \left( A - XP + \lambda_i(M)X\hat{P}_2 \right). \tag{41}
\]

We will prove that \( (A - XP + \lambda_i(M)X\hat{P}_2) \) is a Hurwitz matrix \( \forall i = 1, \ldots, N \), and thus prove the theorem. If \( \lambda_i(M) = 0 \) then \( A - XP + \lambda_i(M)X\hat{P}_2 \) is Hurwitz based on Remark 3. If \( \lambda_i(M) \neq 0 \), then from Corollary 2 and from condition (38), we conclude that \( A - XP + \lambda_i(M)X\hat{P}_2 \) is Hurwitz. \( \square \)

Theorem 3 has several main consequences:

1) If \( M \in K_{m,n}^N(G) \), then \( \hat{K} \) in (39) is an asymptotically stable distributed controller.
2) We can use one local LQR controller to compose distributed stabilizing controllers for a collection of identical dynamically decoupled subsystems.
3) The first two consequences imply that we can not only find a stabilizing distributed controller with a desired sparsity pattern (which is in general a formidable task by itself), but it is enough to solve a low-dimensional problem (characterized by \( d_{\text{max}}(G) \)) compared to the full problem size (36). This attractive feature of our approach relies on the specific problem structure defined in Section II and IV.

4) The eigenvalues of the closed-loop large-scale system \( \mathcal{S}(A_{cl}) \) can be computed through \( N \) smaller eigenvalue computations as \( \bigcup_{i=1}^{N} \mathcal{S} \left( A - XP + \lambda_i(M)X\hat{P}_2 \right) \).

5) The result is independent of the local LQR tuning. Thus \( Q_1, Q_2 \) and \( R \) in (23) can be used in order to influence the compromise between minimization of absolute and relative terms, and the control effort in the global performance.

For the special class of systems defined by Condition 1, the hypothesis of Theorem 3 can be relaxed as follows:

\textbf{Theorem 4:} Consider the LQR problem (10) with weights chosen as in (23) and its solution (29), (31). Assume that Condition 1 holds.

Let \( M \in \mathbb{R}^{N \times N} \) be a symmetric matrix with the following property:

\[
\lambda_i(M) \geq 0, \quad \forall \lambda_i \in \mathcal{S}(M). \tag{42}
\]

Then, the closed loop system (40) is asymptotically stable when \( \hat{K} \) is constructed as in (39).

\textbf{Proof:} Notice that if Condition 1 holds, then \( \mathcal{S} \left( A - XP + \lambda_iX\hat{P}_2 \right) \) is Hurwitz for all \( \lambda_i(M) \geq 0 \) (from Corollary 1 and Corollary 2). By Proposition 2

\[
\mathcal{S} \left( I_N \otimes (A - XP) + M \otimes (X\hat{P}_2) \right) = \bigcup_{i=1}^{N} \mathcal{S} \left( A - XP + \lambda_i(M)X\hat{P}_2 \right),
\]

which together with condition (42) proves the theorem. \( \square \)

In the next sections we show how to choose \( M \) in Theorem 3 and Theorem 4 in order to construct distributed suboptimal controllers. The matrix \( M \) will (i) reflect the structure of the graph \( G \), (ii) satisfy (38) or (42) and (iii) be computed by using the graph adjacency matrix or the Laplacian matrix.

In order to make the exposition simpler, we will start with a simple special graph in Section IV-A and generalize the results in Section IV-B to any graph structure.

\textbf{A. Finite Strings}

Let \( G \) represent a string interconnection of \( N \) systems with \( d_{\text{max}}(G) = 2 \) and the following Laplacian:

\[
L(G) = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix}. \tag{43}
\]

Construct a distributed controller as follows. Solve the LQR problem (10), (23) with \( N_L = 2 + 1 = 3 \) and desired \( Q_1, Q_2, R \). Consider the decomposition (29) and (31) of local controller and cost function, respectively. Take \( b \geq 0 \) and construct the global controller gain matrix \( \hat{K} \) as

\[
\hat{K} = \begin{bmatrix} K_1 & 0 & 0 & \cdots & 0 & aK_2 & bK_2 & 0 & \cdots & 0 \\ 0 & K_1 & 0 & \cdots & 0 & bK_2 & aK_2 & bK_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & K_1 & 0 & 0 & \cdots & bK_2 & aK_2 & bK_2 \\ 0 & \cdots & 0 & 0 & K_1 & 0 & \cdots & 0 & bK_2 & aK_2 \end{bmatrix}. \tag{44}
\]
The closed-loop system under the distributed control law $\hat{K}$ can be written as $\dot{x} = A_d\hat{x}$ with

$$A_d = I_N \otimes A + (I_N \otimes B)\hat{K}. \quad (45)$$

Rewrite $\hat{K}$ as:

$$\hat{K} = -I_N \otimes (R^{-1}B'\hat{P}_1) - aI_N \otimes (R^{-1}B'\hat{P}_2) - bA(\hat{G}) \otimes (R^{-1}B'\hat{P}_2).$$

From Theorem 2, $\hat{K}$ is equal to:

$$\dot{\hat{K}} = -I_N \otimes (R^{-1}B'P) + M \otimes (R^{-1}B'\hat{P}_2)$$

and

$$M = (NL - 1 - a)I_N - bA(\hat{G}), \quad NL = 3. \quad (46)$$

The following corollary of Theorem 3 defines the range of $a$ and $b$ which leads to a stable distributed controller.

**Corollary 3:** Consider controller (44). If $a + 2b < \frac{1}{2}$, then the closed loop system (40) is asymptotically stable. If Condition 1 holds then the closed loop system (40) is asymptotically stable for all $a + 2b \leq 2$.

**Proof:** From Property 1, $\lambda_{\max}(A(\hat{G})) \leq d_{\max}(\hat{G}) = 2$. Since $b \geq 0$, by using Proposition 1 applied to $M$ we derive:

$$\lambda_{\min}(M) = NL - 1 - a - b\lambda_{\max}(A(\hat{G})) \geq NL - 1 - a - 2b, \quad NL = 3. \quad (47)$$

Condition (38) of Theorem 3 holds if $a + 2b < \frac{1}{2}$, which proves the first part of the Theorem.

Assume that Condition 1 holds and consider equation (47). Then, condition (42) of Theorem 4 holds if $a + 2b \leq 2$. $\square$

**Example 1:** With $a = 0$ the distributed controller (44) becomes

$$\hat{K} = \begin{bmatrix} K_1 & bK_2 & 0 & \cdots & 0 \\ bK_2 & K_1 & bK_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & bK_2 & K_1 & bK_2 \\ 0 & \cdots & 0 & bK_2 & K_1 \end{bmatrix}. \quad (48)$$

The closed loop system (40) is asymptotically stable if $b < \frac{1}{2}$. If Condition 1 holds, then the closed loop system (40) is asymptotically stable if $b \leq 1$.

**Example 2:** Assume that Condition 1 holds. Since the graph Laplacian $L(\hat{G})$ is a symmetric positive semidefinite matrix, $M = aL(\hat{G}), \quad a \geq 0$ satisfies condition (42) of Theorem 4. Thus any distributed controller $\hat{K} = -I_N \otimes (R^{-1}B'P) + aL(\hat{G}) \otimes (R^{-1}B'\hat{P}_2)$ having the following structure

$$\hat{K} = \begin{bmatrix} 0 & K_2 & 0 & \cdots & 0 \\ K_2 & 0 & K_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & K_2 & 0 & K_2 \\ 0 & \cdots & 0 & K_2 & 0 \end{bmatrix} + I_N \otimes (K_1 - (2a)K_2)$$

stabilizes system (40) for all $a \geq 0$.

### B. Arbitrary Graph Structures

We consider a generic graph $\hat{G}$ for $N$ nodes with an associated Laplacian $L(\hat{G})$ and maximum vertex degree $d_{\max}$. Let $0 = \lambda_1(\hat{G}) < \lambda_2(\hat{G}) \leq \lambda_N(\hat{G})$ be the eigenvalues of the the Laplacian $L(\hat{G})$. In the following Corollaries 4, 5 and 6 we present three ways of choosing $M$ in (39) which lead to distributed suboptimal controllers.

**Corollary 4:** Compute $M$ in (39) as:

$$M = aL(\hat{G}). \quad (50)$$

If

$$a > \frac{NL}{2\lambda_2(\hat{G})}, \quad (51)$$

then the closed loop system (40) is asymptotically stable when $\hat{K}$ is constructed as in (39). In addition, if Condition 1 holds, then the closed loop system (40) is asymptotically stable for all $a \geq 0$.

**Proof:** Apply condition (38) in Theorem 3 to the case $M = aL(\hat{G})$. Notice that $\lambda_i(\hat{G}) \geq \lambda_2(\hat{G})$ for all $i = 3, \ldots, N$ and $\lambda_2(\hat{G}) \neq 0$ by Property 3 of the Laplacian matrix. Therefore condition (38) becomes $\lambda_2(aL(\hat{G})) > \frac{NL}{2\lambda_2(\hat{G})}$. This implies $a\lambda_2(\hat{G}) > \frac{NL}{2}$ and thus $a > \frac{NL}{2\lambda_2(\hat{G})}$. The first part of the Theorem is proven. If $a \geq 0$ then $\lambda_i(M) \geq 0 \forall i$. Therefore the application of Theorem 4 to the matrix $M = aL(\hat{G})$ proves the second part of the theorem. $\square$

**Remark 5:** By using the third formula in Property 4, condition (51) can be linked to the edge connectivity as follows

$$a > \frac{NL}{2\eta(\hat{G}) \left(1 - \cos \left(\frac{\pi}{N}\right)\right)}. \quad (52)$$

**Remark 6:** Corollary 4 links the stability of the distributed controller to the size of the second smallest eigenvalues of the graph Laplacian. It is well known that graphs with large $\lambda_2$ (with respect to the maximal degree) have some properties which make them very useful in several applications such as computer science [29]–[32]. Interestingly enough, this property is shown here to be crucial also for the design of distributed controllers. We refer the reader to [26] for a more detailed discussion on the importance of the second smallest eigenvalue of a Laplacian.

**Corollary 5:** Compute $M$ in (39) as:

$$M = aI_N - bA(\hat{G}), \quad b \geq 0. \quad (53)$$

If $a - bd_{\max} > \frac{NL}{2}$, then the closed loop system (40) is asymptotically stable when $\hat{K}$ is constructed as in (39). In addition, if Condition 1 holds, then the closed loop system (40) is asymptotically stable if $a - bd_{\max} \geq 0$.

**Proof:** Notice that $\lambda_{\min}(M) = a - b\lambda_{\max}(A(\hat{G})) \geq a - bd_{\max}$. The proof is a direct consequence of Theorems 3 and 4 and Property 1 of the adjacency matrix. $\square$

Consider a weighted adjacency matrix $A^w = A^w(\hat{G})$, defined as follows. Denote by $A^w_{i,j} \in \mathbb{R}$ its $i,j$ element, then $A^w_{i,j} = 0$, if $i = j$ and $(i,j) \notin A$ and $A^w_{i,j} = w_{ij}$ if $(i,j) \in A$, where $w_{ij}$ represents the weight of the edge between nodes $i$ and $j$. The stability of the distributed controller can then be related to the weights of the edges.

$$A^w = A^w(\hat{G}), \quad A \subseteq A^w \quad (54)$$

Therefore, if $\lambda_{\min}(M) = a - b\lambda_{\max}(A(\hat{G})) \geq a - bd_{\max}$, then the closed loop system (40) is asymptotically stable when $\hat{K}$ is constructed as in (39). In addition, if Condition 1 holds, then the closed loop system (40) is asymptotically stable if $a - bd_{\max} \geq 0$.
\( \forall i, j = 1, \ldots, N, i \neq j \). Assume \( w_{ij} = w_{ji} > 0 \). Define \( w_{\text{max}} \) as

\[
\text{max}_{i} w_{ij} = \max_{i} \left\{ \sum_{j} w_{ij} \right\}
\]

**Corollary 6:** Compute \( M \) in (39) as:

\[
M = aI_{N} - A^{w}(G).
\]

If \( a > w_{\text{max}} - \frac{N}{b} \), then the closed loop system (40) is asymptotically stable when \( \tilde{K} \) is constructed as in (39). In addition, if Condition 1 holds, then the closed loop system (40) is asymptotically stable if \( a \geq w_{\text{max}} \).

**Proof:** \( A^{w}1 \leq w_{\text{max}}1 \) and by Perron-Frobenius Theorem \( \lambda_{\text{max}}(A^{w}) \leq w_{\text{max}} \). Notice that \( \lambda_{\text{min}}(M) = a - \lambda_{\text{max}}(A(G)) \geq a - w_{\text{max}} \), then the proof is a direct consequence of Theorems 3 and 4. \( \square \)

The results of Corollaries 4-6 are summarized in Table I.

<table>
<thead>
<tr>
<th>Choice of ( M )</th>
<th>Stability Condition</th>
<th>Stability Condition If Condition 1 Holds</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M = aI_{N} - bA(\tilde{G}) )</td>
<td>( a &gt; \frac{N}{b} )</td>
<td>( a \geq 0 )</td>
</tr>
<tr>
<td>( aI_{N} - A^{w}(\tilde{G}) )</td>
<td>( a &gt; w_{\text{max}} - \frac{N}{b} )</td>
<td>( a \geq w_{\text{max}} )</td>
</tr>
</tbody>
</table>

**Table 1**

**Summary of stability conditions in Corollaries 4-6 for the closed loop system (39)-(40).**

Corollaries 4-6 present three choices of distributed control design with increasing degrees of freedom. In fact, \( a, b \) and \( w_{ij} \) are additional parameters which, together with \( Q_{1}, Q_{2} \) and \( R \), can be used to tune the closed-loop system behavior. We recall here that from Theorem 3, the eigenvalues of the closed-loop large-scale system are related to the eigenvalues of \( M \) through the simple relation (41). Thus as long as the stability conditions defined in Table I are satisfied, the overall system architecture can be modified arbitrarily by adding or removing subsystems and interconnection links. This leads to a very powerful modular approach for designing distributed control systems. The difference between the performance of the distributed control law and the performance of a centralized optimal solution can be also computed in a simple way. This is discussed in the next section. We remark that detailed performance analysis of the proposed distributed control design, or the design of distributed control laws with performance guarantees are not discussed within this manuscript and represent future research topics.

**C. Measure of Suboptimality**

Consider system (35) and let \( K^{*} \) and \( P^{*} \) be an LQR controller and the corresponding ARE solution associated with the weights \( \bar{Q} > 0 \) and \( \bar{R} > 0 \). Therefore, \( \hat{u}(t) = -K^{*}\hat{x}(t) \) minimizes the following cost function for any \( \hat{x}_{0} \):

\[
J^{*}(\hat{x}_{0}) = \min_{\hat{u}(t)} \| \Delta P \|
\]

subject to

\[\dot{\hat{x}} = \bar{A}\hat{x} + \bar{B}\hat{u}, \quad \hat{u}(t) = K_{d}\hat{x}(t), \quad K_{d} \in \mathbb{K}_{m,n}(G), \quad \hat{x}(0) = \hat{x}_{0} \]

The “best” linear state-feedback distributed controller \( K_{d} \in \mathbb{K}_{m,n}(G) \) could be computed by solving:

\[
J_{d}^{*}(\hat{x}_{0}) = \min_{K_{d}} \| \Delta P \|
\]

subject to

\[\dot{\hat{x}} = \bar{A}\hat{x} + \bar{B}\hat{u}, \quad \hat{u}(t) = K_{d}\hat{x}(t), \quad K_{d} \in \mathbb{K}_{m,n}(G), \quad \hat{x}(0) = \hat{x}_{0} \]

As discussed in the previous sections, computing the solution to problem (61), is a difficult problem in general without further assumptions. Its efficient solution is the topic of current research.
Consider a $10 \times 10$ mesh interconnection of $N = 100$ identical, dynamically decoupled and independently actuated linear systems moving in a plane with double integrator dynamics in both spatial dimensions:

$$\ddot{x}_i = u_{x,i}, \quad \ddot{y}_i = u_{y,i}, \quad i = 1, \ldots, 100$$

The interconnection structure is depicted in Figure 1. The control objective is to move each subsystem to a desired absolute rectangular grid, which has equal separation distances defined between each orthogonal neighbor.

![Fixed mesh grid](image)

**Fig. 1.** The $10 \times 10$ finite mesh grid interconnection structure of the simulation example.

The maximum vertex degree of such an interconnection graph is 4, as the nodes located at the “corners” of the rectangular grid have 2 neighbors, those along the “edges” of the rectangle have 3 neighbors and the ones in the “middle” have 4 each.

A stabilizing distributed controller is designed for the mesh interconnection of systems, by solving the following, simple LQR problem involving only $N_L = 5$ nodes ($d_{\text{max}} + 1$).

$$\min_{\hat{u}} J(\hat{u}, \hat{x}_0)$$

$$\text{subj. to } \dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}\hat{u}$$

$$\hat{x}(0) = \hat{x}_0$$

where

$$\hat{A} = I_5 \otimes A, \quad \hat{B} = I_5 \otimes B,$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}.$$  

(64)

The cost function defined in (5) uses the following weights for absolute and relative state information

$$Q_{ii} = Q_a, \quad Q_{ij} = Q_r, \quad \forall i \neq j.$$  

(65)

The weight on the control effort was kept constant in the following examples ($R = 1$ for each control input in both spatial dimensions). The subsystem model and chosen weights satisfy Condition 1 (we tested the stability of (32) by solving an LMI problem searching for quadratic parameter-dependent Lyapunov functions), thus we can use less conservative stability ranges in Table I. Alternatively, one can choose the more conservative stability ranges in Table I which can be always satisfied.

The solution of the above local LQR problem will yield a controller matrix of the following form in this specific example:

$$\hat{K} = \begin{bmatrix} K_a & K_r & K_r & K_r \\ K_r & K_a & K_r & K_r \\ K_r & K_r & K_a & K_r \\ K_r & K_r & K_r & K_a \end{bmatrix}.$$  

(66)

The distributed controller for the grid of $N = 100$ interconnected systems has the same structure as the underlying graph and is constructed in the following simple way based on results presented in Section IV-B:

$$\bar{K} = I_N \otimes K_a + A \otimes K_r,$$  

(67)

where $A$ denotes the adjacency matrix of the mesh grid interconnection of systems. The distributed controller in (67) can be rewritten as $\bar{K} = I_N \otimes K - (4I_N - A) \otimes K_r$, where $K = K_a + 4K_r$. This corresponds to $M = 4I_N - A$ in Table I with stability condition $a - d_{\text{max}} \geq 0$, which is satisfied in this example since $a = d_{\text{max}} = 4$.

Two simulation results, starting from the same initial conditions, are presented with different choices for the parameters $Q_a$ and $Q_r$ in the local LQR problem tuning (65). The first simulation weights the absolute state information in the local LQR solution more heavily than the second simulation and uses $Q_a(1, 1) = 1$ and $Q_r(1, 1) = 1$ values for position states in both spatial dimensions. Velocity states were not weighted in these simulation examples. Snapshots of the simulation are shown in Figure 2, indicating that the subsystems converge to their desired absolute positions along fairly straight lines starting from their initial conditions. The second simulation illustrates the behavior of the large-scale distributed control system, when much more emphasis is put on the relative state information. As the snapshots in Figure 3 demonstrate, the behavior of the overall interconnected system has changed fundamentally. Although the distributed controller is still stabilizing, the dynamic behavior has changed drastically and shows wave-like oscillations due to the high weights on relative state information.

Videos of these simulation examples can be found at [33]. The purpose of these examples was to show with a numerical exercise that the stabilizing effect of the proposed distributed control design is independent of the local LQR weighting parameter selection, thus these parameters are freely available for tuning and achieving different global performance objectives.

**VI. Concluding Remarks**

We have introduced a stabilizing distributed control design method for large-scale interconnections of identical, dynamically decoupled and independently actuated linear systems. The procedure requires the solution of a single local LQR
problem, whose size depends only on the maximum vertex degree of the interconnection graph. Special properties of the local LQR problem were derived, which enable the construction of stabilizing distributed controllers independently of the choice of weighting matrices. The relationship between stability of the overall large-scale system, the robustness of local controllers and the spectrum of a sparsity pattern matrix has been highlighted. The proposed distributed controller will be stabilizing even if subsystems and communication links are added to or removed from the overall system, as long as this does not change the maximum vertex degree or violate the conditions given in Table I. This feature provides a very powerful modularity to our approach.

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