Sufficient Conditions for Complexity Reduction in Min-Max Control of Constrained Uncertain Linear Systems

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Abstract—In this paper we consider finite-time min-max optimization problems for linear systems with additive disturbances subject to hard input constraints and soft state constraints.

We present a set of state-dependent sufficient conditions which allow for the efficient solution of the min-max optimization problem. The set of initial states for which the proposed conditions are satisfied is computed by solving a sequence of linear programs.

We compare the proposed method to open-loop, vertex enumeration, and affine disturbance feedback techniques in terms of computational complexity and optimal cost using a simple example. We also demonstrate the efficacy of the approach in predictive control design for radiant-slab cooling systems.

I. INTRODUCTION

In this paper we consider finite-time min-max optimization problems for linear systems with additive disturbance subject to hard input constraints and soft state constraints.

We consider a cost composed of a weighted sum of 1- and $\infty$- norms on the input and constraint violation vectors. The cost considered does not need to be separable. This is useful, for instance, in energy control problems where peak power (the infinity norm of the entire input sequence over the control horizon) has to be minimized.

The objective of this paper is to compute the solution over feedback policies that satisfies input constraints and minimizes the worst case cost for all admissible disturbances. The solution is then used for implementation in min-max robust model predictive control (MPC). We do not impose any structure on the feedback policies.

Min-max robust MPC for linear systems was first investigated by Witsenhausen [1] and Campo and Morari [2]. In general, the feedback solution is obtained either by dynamic programming or by solving one optimization problem (batch approach) considering the scenario of all possible disturbance realizations [1], [3]–[12]. Using dynamic programming, min-max robust control of linear systems was investigated by Lee and Yu [9] for uncertain parameters residing in ellipsoids and polyhedrons. Offline computation of state feedback policies for linear systems with additive and parametric uncertainties was investigated by Bemporad, Borrelli and Morari [13] using multi-parametric programming detailed in [14] for linear programs and [15]–[17] for quadratic programs. However, dynamic programming in this paper is challenged by the non-separability in time of the cost, while the cost is assumed to be separable in the cases discussed above. Robust min-max optimization for linear systems with additive disturbance was investigated by Scokaert and Mayne [8] using a single optimization with vertex enumeration. However, the computation time for vertex enumeration is exponential with respect to the horizon length, rendering the solution method intractable for long prediction horizons.

Tractable alternatives to computing exact feedback solutions to the min-max problem include using open-loop input sequences [18] and affine disturbance feedback [19]. However, in general using open-loop input sequences lead to conservative solutions [3]. Goulart, Kerrigan, and Maciejowski [19] detail the use of affine disturbance feedback in the robust control of linear systems with additive disturbance. The solution of the min-max problem using affine disturbance feedback subject to robust linear constraints was addressed by Oldewurtel [20] in the context of stochastic MPC. While affine disturbance feedback is computationally efficient, it is conservative because, in general, the optimal feedback policies are non-linear.

In this paper, we present a set of state-dependent sufficient conditions which allow for the computation of the exact solution of the min-max problem over all feedback policies without using dynamic programming and yet avoiding vertex enumeration. The idea of this paper is to predetermine a set of disturbance realizations which, for some initial states and inputs, worst case performance is attained. Then, we compute the set of initial states for which the worst case performance is attained for all admissible inputs at the precomputed set of disturbance realizations. Our experience with building energy control systems show that this approach may be very effective [21], [22]. The computational complexity of the proposed method is linear with respect to the length of the control horizon.

We compare the proposed method to open-loop, vertex enumeration, and affine disturbance feedback techniques [8], [19], [20] in terms of computational complexity and optimal cost using a simple example. We then demonstrate the efficacy of the approach in an implementation of model predictive control for radiant-slab cooling systems.

II. PROBLEM DEFINITION

Consider the linear time-invariant discrete-time system

$$x_{t+1} = Ax_t + Bu_t + d_t,$$  \hspace{1cm} (1)
where $x_t \in \mathbb{R}^n$ is the system state, $u_t \in \mathbb{R}^p$ the controlled input, $d_t \in \mathbb{R}^n$ the disturbance, $A \in \mathbb{R}^{n \times n}$, and $B \in \mathbb{R}^{n \times p}$. The system is subject to the constraints

$$u_t \in U, \ \forall t \geq 0 \ \text{and} \ \forall t \geq 0,$$

(2)

where $U \subset \mathbb{R}^p$ and $D \subset \mathbb{R}^n$ are polytopes for all $t \geq 0$.

Consider a fixed control horizon length $N$. We define

$$A = \begin{bmatrix} I_m \\ A \end{bmatrix}, \quad B = \begin{bmatrix} 0_{m \times n} & \cdots & 0_{m \times n} \\ B & \cdots & \vdots \\ \vdots & \cdots & 0_{m \times n} \\ A^{N-1}B & \cdots & AB & B \end{bmatrix},$$

(3)

and

$$W = \begin{bmatrix} u_t \\ u_{t+1} \\ \vdots \\ u_{t+N-1} \\ x_t \\ x_{t+1} \\ \vdots \\ x_{t+N} \end{bmatrix},$$

so that we can compactly rewrite the evolution of system (1) as

$$x_t = Ax_t + Bu_t + Wd_t.$$

We also define

$$U = U \times \cdots \times U \quad \text{and} \quad D = D \times \cdots \times D,$$

(4)

Let $P \in \mathbb{R}^{m \times Nn}$, $b \in \mathbb{R}^m$. We assume the system (1) to be subject to the following soft state constraints over the control horizon $[0, N]$

$$Px_t \leq b,$$

where the inequalities are considered element-wise.

For a given state sequence $x_t$, the violation of the $i$-th constraint in (4) is

$$\delta_i(P, b, x_t) = \max \{0, P_i x_t - b_i \},$$

where $P_i$ and $b_i$ are the $i$-th rows of the matrix $P$ and vector $b$, respectively.

The violation vector, $\delta(P, b, x_t)$, is then defined as

$$\delta(P, b, x_t) = [\delta_1(P, b, x_t), \ldots, \delta_m(P, b, x_t)].$$

A. Min-Max Finite-Time Optimal Control

At each time $t \geq 0$, we are given a reference input signal $\bar{u}_k \in \mathbb{R}^p$ for $k \in \{t, \ldots, t + N - 1\}$. We define

$$\bar{u}_t = \begin{bmatrix} \bar{u}_t & \bar{u}_{t+1} & \cdots & \bar{u}_{t+N-1} \end{bmatrix}^T.$$
We omit the proof to Proposition 1 since we will prove a generalization of it in the next proposition. If a single disturbance realization causes the largest constraint violation at a subset of the constraints, and all other constraints are never violated, then we can still solve problem (6) considering only that disturbance realization. This is stated in the proposition below.

**Proposition 2:** Suppose there is a subset $S$ of the set $\{1, \ldots, m\}$ such that

1) $\exists \alpha^* \in A$ that is a solution to

$$\arg \max_{\alpha \in A} \; P_t \cdot W \hat{d}_t^\alpha$$

for all $i \in S$.

2) $P_t \cdot A x_t + P_t \cdot B u_t^\alpha + P_t \cdot W \hat{d}_t^\alpha - b_t \leq 0 \; \forall \alpha \in A, u_{j,t}^\alpha \in U, j \geq t, i \in \{1, \ldots, m\} \setminus S$.

Then, problem (6) is equivalent to problem (7).

**Proof:** Let the optimal cost to problem (6) be $C_1$ and the optimal cost to problem (7) be $C_2$. It is easy to see that $C_1 \geq C_2$. Since $P_t \cdot W \hat{d}_t^\alpha \leq P_t \cdot W \hat{d}_t^\alpha \forall \alpha \in A, i \in S$, we have

$$P_t \cdot A x_t + P_t \cdot B u_t^\alpha + P_t \cdot W \hat{d}_t^\alpha - b_t \leq 0 \; \forall \alpha \in A, i \in S.$$
4) Suppose there are \( r > 1 \) partitions. For each \( i \in \{1, \ldots, m\} \), solve the linear program

\[
\arg\max_{w \in \mathcal{W}} \mathbf{p}_i \mathbf{u}_i
\]

and store the cost \( C^{*,i} \).

5) For \( k \in \{1, \ldots, r\} \), compute the polytope of initial states, \( \mathcal{X}^k \), defined by the set

\[
\mathcal{X}^k = \{ x_t \in \mathbb{R}^n | \mathbf{p}_i \mathbf{A} x_t + C^{*,i} + D^{*,i} - b_i \leq 0, \quad i \in \{1, \ldots, m\} \setminus S_k \}
\]

Then, for each \( k \in \{1, \ldots, r\} \), \( \alpha^{*,k} \), along with the states \( x_t \in \mathcal{X}^k \), satisfy the conditions of proposition (2). Therefore, \( \bigcup_{k=1}^{r} \mathcal{X}^k \) are the set of initial states for which proposition (2) can be applied.

Note that if \( D \) and \( U \) are hypercubes, then the linear programs in steps (1) and (4) are easily solved. To illustrate this, consider the linear program

\[
\max_{y \in \mathcal{Y}} c^T y_t
\]

where \( \mathcal{Y} \subseteq \mathbb{R}^n \) is a hypercube defined by \( \mathcal{Y} = [y_{min,1}, y_{max,1}] \times \cdots \times [y_{min,n}, y_{max,n}] \). We write \( c^T = \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix} \). Then, the optimizer for problem (9) is \( y^* = \begin{bmatrix} y_{1}^* & \cdots & y_n^* \end{bmatrix} \), where

\[
y_{i}^* = \begin{cases} y_{max,i} & \text{if } c_i \geq 0 \\ y_{min,i} & \text{if } c_i < 0 \end{cases} \quad \forall i \in \{1, \ldots, n\}.
\]

III. NUMERICAL EXAMPLE

A. Efficient batch closed-loop vs. alternative methods

Consider the 1-state, 1-input system described by

\[
x_{t+1} = x_t + u_t + d_t
\]

subject to the constraints \( u_t \in [-1, 1] \) and \( d_t \in [-3, 3] \). Let

\[
\mathbf{p} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 13 \\ 13 \\ 13 \\ 13 \\ \vdots \\ 13 \end{bmatrix}, \quad (11)
\]

where \( \mathbf{p} \in \mathbb{R}^{2(N+1) \times N+1} \) and \( \mathbf{b} \in \mathbb{R}^{2(N+1)} \) corresponds to desired box constraints on the system state.

Over a fixed control horizon \( N \), consider solving the problem (6) at \( t = 0 \) with norm on the input satisfying A4 with \( \rho = 1 \) and \( f(\delta) = \|\delta\|_1 + \|\delta\|_\infty \). This problem satisfies A1-A6. Therefore, we may apply Proposition 3.

We compare our proposed method to the vertex enumeration method, affine disturbance feedback, and open-loop prediction for the system described above. We tabulate the solution cost for different values of \( x_0 \) and for \( N = 7 \) in the table below.

The proposed method and vertex enumeration methods achieved the same costs because \( d_{\text{max}} \) and \( d_{\text{min}} \) are vertices of the disturbance set. Open-loop predictions and affine disturbance feedback returned higher costs than the two for most of the initial states investigated. The difference between vertex enumeration and the other methods is apparent when we investigate the mean computation time for the methods.

To see this, we fixed \( x_0 = 0 \) and recorded the computation time in seconds for \( N = 2, 3, 4, 5, 6 \) in the table below.

<table>
<thead>
<tr>
<th>Horizon length ( N )</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proposed method</td>
<td>0.2107</td>
<td>0.1478</td>
<td>0.1492</td>
<td>0.1506</td>
<td>0.1569</td>
</tr>
<tr>
<td>Vertex enumeration</td>
<td>0.1622</td>
<td>0.2060</td>
<td>0.2956</td>
<td>0.5128</td>
<td>0.9698</td>
</tr>
<tr>
<td>Open-loop</td>
<td>0.1671</td>
<td>0.1491</td>
<td>0.1515</td>
<td>0.1563</td>
<td>0.1544</td>
</tr>
<tr>
<td>Affine feedback</td>
<td>0.1942</td>
<td>0.2291</td>
<td>0.2613</td>
<td>0.2971</td>
<td>0.3349</td>
</tr>
</tbody>
</table>

Table II

COMPUTATION TIME AS FUNCTION OF HORIZON LENGTH, \( x_0 = 0 \)

Vertex enumeration is the slowest of the four methods, which is expected since its complexity is exponential with respect to the horizon length. Affine disturbance feedback, open-loop predictions, and the proposed method had comparable computation times, with affine disturbance feedback only slightly slower than the other two.

We compare the computational times of open-loop prediction, affine disturbance feedback, and our proposed method for a horizon length of \( N = 24 \) for \( x_0 = 0 \), for which the vertex enumeration method is computationally intractable. For our proposed method, the computation time was 0.1952 seconds. For open-loop predictions, the computation time was 0.1938 seconds. Using affine disturbance feedback, the computation time was 1.1850 seconds. Our proposed method and open-loop predictions are noticeably faster than affine disturbance feedback, which requires more variables than either of the other two methods. Our proposed method again achieved the lowest cost of the three methods.

B. Radiant slab HVAC systems

We apply our results to the energy efficient control of the radiant slab system at the Brower Center in Berkeley, CA. For this example, we consider the MPC of a single radiant slab zone. We define the thermal model of the system as follows:

\[
T_t = \begin{bmatrix} T_{\text{slab},t} & T_{\text{room},t} \end{bmatrix}^T
\]

represents the states of the system, where \( T_{\text{slab}} \) is the temperature of the radiant slab and \( T_{\text{room}} \) is the temperature of the room. \( u_t \) is the temperature of the water supplied to the radiant slab. The radiant slab system can be approximated by a linear system update equation of the form

\[
T_{t+1} = AT_t + Bu_t + Wd_t,
\]

where

\[
A = \begin{bmatrix} 0.9579 & 0.0406 \\ 0.0093 & 0.9883 \end{bmatrix}, \quad B = \begin{bmatrix} 0.0016 \\ 0 \end{bmatrix}, \quad W = \begin{bmatrix} 0 \\ 0.0025 \end{bmatrix}.
\]
and $d_t$ is the outside air temperature at time $t$, with time measured in hours. The parameters in equation (12) were identified by performing step-tests on the actual building.

We are interested in controlling the water temperature supplied to the slabs to maintain the room air temperature within acceptable bounds. Consider a control horizon $N = 24$ hours and suppose we wish to keep the room air temperature between $65^\circ F$ and $75^\circ F$ at all times within the horizon. We define

$$
P = \begin{bmatrix}
0 & 1 & \cdots & \cdots & \cdots & 0 \\
0 & -1 & \cdots & \cdots & \cdots & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 \\
0 & 0 & 0 & -1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 0 & 1 \\
0 & \cdots & \cdots & 0 & 0 & -1 \\
\end{bmatrix}, \quad b = \begin{bmatrix} 75 \\ \vdots \\ 75 \\ -65 \end{bmatrix}, \quad (13)
$$

where $P \in \mathbb{R}^{50 \times 50}$ and $b \in \mathbb{R}^{50}$.

For the system considered, the supply water temperature is constrained to be within $55^\circ F$ and $90^\circ F$. We investigate controlling the building temperature on a hot summer day, with a 48-hour outside air temperature prediction, $OAT_t$, as shown in Figure (1).

We now proceed with the RHC of the radiant slab system. Suppose that we start at the initial state $x_0 = [70 \ 66]^T$. At each time $t = 0, ..., 23$, we check that the state, $x_t$, is in $X_t$ and then solve problem (7) in order to compute the control sequence $u_t^*$. Then we apply the first control input $u_0^*$ and resolve problem (7) at time $t + 1$. We ran the RHC simulation for three different test cases: the nominal disturbance profile, the maximal (hottest) disturbance profile, and the minimal (coldest) disturbance profile. In closed-loop, the state $T_t \in X_t$ for all $t \in \{1, ..., 24\}$. We report the plots of the room temperature for these three test cases in Figure 3.

The results for the closed-loop room air temperature are as expected. The plot suggests that as the outside air temperature is increased, the closed-loop room air temperature increases as well. Figure (3) shows the closed-loop input water temperature for the three cases. The plot shows that when the maximal disturbance profile is applied, the input water temperature is always saturated at $55^\circ F$. However, if the nominal disturbance profile is applied, the input water temperature switches to the nominal zero-cost value of $70^\circ F$ at certain times of the day. When the minimal disturbance profile is applied, the input water temperature switches to $70^\circ F$ for an even larger period of the day.
and successfully kept the room temperature within tolerable bounds. We then demonstrated the efficacy of the approach in an implementation of model predictive control: stability and optimality,” vol. 36, pp. 789–814, 2000.


