Decentralized Robust Control Invariance for a Network of Storage Devices

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Abstract—Robust control of networked storage devices is considered. Each storage device is modeled as a single-state, discrete–time integrator with bounded control input, and subject to additive disturbance. The values of the disturbance are unknown but are assumed to belong to bounded sets. Nodes exchange stored resource through links of limited capacity. Characterization of the maximal robust control invariant (RCI) set is provided and decentralization of the robust feasibility problem is considered. A notion of decentralized RCI set is introduced and a parametrization of a family of decentralized RCI sets is proposed. It is shown that the set of parameters for which the decentralized RCI sets are non-empty is a polyhedral set. The result offers a possibility to use convex optimization for selection of bounds on the admissible flow through the network which ensure feasibility of the proposed decentralized design.

I. INTRODUCTION

Networks of integrators are models commonly used to characterize a variety of large scale systems. These include networked devices which can store a certain amount of resource (such as water, energy, goods, data), “produce” it and transfer it to its peers. Some problems in production–distribution systems, communication networks, water supply systems, energy and transportation networks can be modeled within such framework.

In this paper we adopt a model where each node of the network represents a discrete–time single–state integrator whose state corresponds to the amount of resource stored within the node. The amount of resource is regulated by the control input associated to the node. Additionally, each node can transfer an amount of its stored resource to other nodes and, equally, receive some amount of resource from other nodes through directed links. The dynamics of the integrator system in each node is affected by an unknown bounded disturbance in the form of resource demand or supply. In such setting the control problem consists of finding feasible productions for each node and flows between the nodes in order to satisfy uncertain demand and keep the state of each node within the admissible bounds and according to some optimality criterion.

The closely related problem of flows in static networks has been treated extensively, starting from early classical works [1], [2] to more recent monographs [3]–[5]. In [1]–[3], [5] the nodes are not dynamical systems but represent merely topological elements. Robust control of dynamic networks subject to uncertain demands is treated in a series of papers by Bianchini and co–authors [6]–[10], both for the discrete–time as well as for continuous–time case, where the disturbances are modeled as non–stochastic and of bounded magnitude. The approach taken in [6]–[8], [11] is to characterize a robust control invariant (RCI) set, i.e., the set of network states for which there exists a network flow that guarantees the demand satisfaction at all time instants. In [6] the conditions for the existence of the RCI set are used for the purpose of optimal network design of production–distribution system in order to achieve feasibility of flows and levels of stored goods in the network for all possible realizations of the demand, while minimizing the cost associated to the network operation. Cost minimization is also addressed in [11], with the focus on existence of the control law that drives the amount of stored goods to the least feasible storage levels for all demands and in presence of failures. An extension earlier results to networks featuring delays in the flows (the control inputs) is provided in [7]. Specific combinatorial structure of maximal RCI sets was hinted on in [6] using the formalism of sub–modular functions and base polyhedra (cf. [5]). Using the projection algorithm the explicit expressions for RCI and positive robust invariant set for a particular control policy are derived in [12] for a specific buffer network structure.

In this paper we do not assume any specific graph structure and consider flow capacities and state constraints more general than those considered in [6], [7], [11]. In the first part of the paper we characterize the maximal robust control invariant set for the defined scenario. In the second part, motivated by the combinatorial structure of the maximal RCI set and computational difficulties that may arise in their applications for large networks, we consider a decentralized design, where the nodes or groups of nodes act as decision makers which treat the flow from adjacent nodes as an additional bounded disturbance. We introduce the concept of decentralized RCI set and propose a parametrization of a family of such sets in the parameters that define constraints in the network. It is demonstrated that the set of parameters for which a non–empty decentralized RCI set exists is a polyhedral set for the considered particular type of polyhedral constraints. Finally, we show how the proposed concept of parametrized decentralized RCI sets offers a way to approach decomposition of a centralized control problem from a set–theoretic perspective, in contrast to more frequently encountered optimization–centric approaches to the problem.

The structure of the paper is the following. The problem statement and necessary preliminaries are given in Section II. We discuss the characterization of the maximal RCI set for the integrator network in Section III. Section IV contains details related to decentralized optimization for selection of bounds on the admissible flow through the network which ensure feasibility of the proposed decentralized design. Numerical example is given in Section V, followed by concluding remarks in Section VI.

Notation and nomenclature: The set of real numbers is denoted by \( \mathbb{R} \) and the set of non–negative (greater than or equal to 0) real numbers as \( \mathbb{R}_+ \). \( \mathbb{N} \) is the set of natural numbers, i.e., \( \mathbb{N} := \{1, 2, \ldots\} \). \( \mathbb{N}_q \) stands for \( \{q_1, \ldots, q_2\} \), where \( q_1, q_2 \in \mathbb{N} \) and \( q_1 \leq q_2 \). Empty set is denoted as \( \emptyset \). Given a finite set \( S \), for any \( S \subseteq \mathbb{N} \) we use notation \( S' := \bar{S} \forall S \), for the set of all \( i \in \mathbb{N} \) such that \( i \notin S \). Given any set \( S \), \( 2^S \) denotes the set of all subsets of \( S \) (power set of \( S \)). Cardinality of a set \( S \) is denoted as \( |S| \). For a vector \( x \in \mathbb{R}^n \) the ith component is denoted as \( x_i \) and \( x(S) := \sum_{i \in S} x_i \) for some \( S \subseteq \mathbb{N}_1\m N \), with convention \( x(\emptyset) = 0 \). Given a set \( S \subseteq \mathbb{R}^n \), \( -S := \{ -s : s \in S \} \). Binary operation \( \oplus \) denotes Minkowski sum of two non–empty sets \( X \) and \( Y \): \( X \oplus Y := \{ x + y : x \in X, y \in Y \} \). Binary operation \( \ominus \) denotes Minkowski (Pontryagin) difference between two non–empty sets \( X \) and \( Y \): \( X \ominus Y := \{ z : z + y \in X, \forall y \in Y \} \). \( \mathbb{R}^n \) denotes, for (nonempty) \( S \subseteq \mathbb{N}_1\m N \), the subspace of \( \mathbb{R}^n \): \( \mathbb{R}^S := \{ x \in \mathbb{R}^n : x_i = 0 \text{ for } i \in \mathbb{N}_1\m N \} \). Given \( X \subseteq \mathbb{R}^n \) and a (non–empty) \( S \subseteq \mathbb{N}_1\m N \), orthogonal projection to the subspace \( \mathbb{R}^S \) is denoted as \( P_X \). Set \( P \subseteq \mathbb{R}^n \) is a polyhedron if it can be represented as \( P = \{ x \in \mathbb{R}^n : h_j^T x \leq k_j, j \in \mathbb{N}_1\m N, k_j \in \mathbb{R}, h_j \in \mathbb{R}^n \} \). Bounded polyhedron is referred to as polytope. Given a polyhedron \( P = \{ x \in \mathbb{R}^n : h_j^T x \leq k_j, j \in \mathbb{N}_1\m N \} \), an inequality \( h_j^T x \leq k_j, j \in \mathbb{N}_1\m N, k_j \neq 0 \), is redundant for \( P \) if \( P = \{ x \in \mathbb{R}^n : h_i^T x \leq k_i, i \in \mathbb{N}_1\m N \} \).

II. PROBLEM STATEMENT AND PRELIMINARIES

Consider a directed graph \( G = (\mathcal{N}, \mathcal{E}) \) where \( \mathcal{N} = \mathbb{N}_1\m N \) is the set of nodes and \( \mathcal{E} \subseteq \mathcal{N} \times \mathcal{N} \) is a set of ordered pairs, edges, whose order indicates the links direction between two nodes. We use the
following notation for adjacent nodes of the node $i \in \mathcal{N}$:
\[ A_i^- = \{ j \in \mathcal{N} : (i, j) \in \mathcal{E} \}, \quad A_i^+ = \{ j \in \mathcal{N} : (j, i) \in \mathcal{E} \}, \quad \text{and} \quad A_i = A_i^- \cup A_i^+ \].

Each node $i \in \mathcal{N}$ is associated with a discrete–time dynamical system with the state update equation:
\[ x_i^+ = x_i + \phi_i - d_i, \tag{1} \]
where $x_i^+$ denotes the state $x_i$ at the next time step. The state $x_i \in \mathbb{R}$ represents the current amount of resource stored in node $i$. The value of $x_i$, controlled by the total net flow $\phi_i$ into the node $i$ is given as:
\[ \phi_i = u_i + \phi_i^+ - \phi_i^-, \tag{2} \]
where $u_i$ represents resource which can be added to or subtracted from the node $i$ from an external source or sink, and $\phi_i^+$ and $\phi_i^-$ denote the amount of resource supplied to / taken from the node $i$ by other nodes in the graph, respectively. The dynamics of each storage element is additionally affected by an unknown disturbance $d_i$ which can be thought of as uncontrolled demand or supply associated to other nodes in the graph, respectively. The dynamics of each storage $i$ is given by:
\[ f_{ij} := u_{ij} - u_{ji}, \tag{3} \]
Note that using these conventions we have that $f_{ij}$ and $f_{ji}$ in (2) are then given as:
\[ \phi_i^+ = \sum_{j \in A_i^+} u_{ij}, \quad \phi_i^- = \sum_{j \in A_i^-} u_{ji}. \]

We further introduce the net flow from the node $i$ to the node $j$ as a difference between the amount of resource transferred from $i$ to $j$ and the amount transferred from $j$ to $i$:
\[ f_{ij} := u_{ij} - u_{ji}. \]
We use the following notation for the net flow between the two disjoint sets $\mathcal{S}, \mathcal{T} \subseteq \mathcal{N}$:
\[ f_{\mathcal{S}, \mathcal{T}} := \sum_{i \in \mathcal{S}, j \in \mathcal{T}} f_{ij}. \]

The dynamics of the network consisting of dynamical systems (1) can be compactly written as:
\[ x^+ = x + \phi - d, \tag{4} \]
where $x = [x_1 \ldots x_n]^T$, $\phi = [\phi_1 \ldots \phi_n]^T$ and $d = [d_1 \ldots d_n]^T$.

Constraints on the state $x$, the flow vector $\phi$ and the disturbance $d$ are given as:
\[ x \in \mathcal{X}, \quad \phi \in \mathcal{F}, \quad d \in \mathcal{D}, \tag{5} \]
where the sets $\mathcal{X}, \mathcal{F}$ and $\mathcal{D}$ are some bounded subsets of $\mathbb{R}^n$. At this point, we do not impose any additional assumptions on the sets $\mathcal{X}, \mathcal{F}$ and $\mathcal{D}$.

Next, we recall some basic terms and results related to robust control invariance. For more in–depth treatment of this rather mature field, the reader is referred to the review [13] and the recent monograph [14].

**Definition 2.1 (RCI set):** A set $\mathcal{R}$ is a robust control invariant (RCI) set for the dynamical system (4) subject to constraints (5) if $\mathcal{R} \subseteq \mathcal{X}$ and for any $x \in \mathcal{R}$ there exists a control (a flow vector) $\phi \in \mathcal{F}$ such that $x + \phi - d \in \mathcal{R}$ for all $d \in \mathcal{D}$. Characterization of RCI sets for constrained and uncertain system (4) represents a fundamental part of constrained, robust control synthesis, such as model predictive control (MPC) (cf. [14], [15]). It is usually of interest to identify the maximal RCI set, which we denote as $\mathcal{R}_\infty$, i.e., the RCI set that is a superset of all RCI sets for a given system and constraints.

Consider the mapping $\rho : 2^\mathcal{X} \to 2^\mathcal{X}$ defined, for a nonempty $\mathcal{S} \subseteq \mathcal{X}$, as follows:
\[ \rho(\mathcal{S}) := \{x \in \mathcal{S} : \exists \phi \in \mathcal{F} \text{ s.t. } x + \phi - d \in \mathcal{S}, \forall d \in \mathcal{D}\}. \tag{6} \]

The mapping $\rho(\cdot)$ gives the set of all states belonging to the argument set that can be “robustly steered” into the same set. From the Definition 2.1 and the expression for $\rho(\cdot)$ it is evident that a nonempty RCI set $\mathcal{R} \subseteq \mathcal{X}$ is a fixed point of the mapping $\rho(\cdot)$, i.e., that $\rho(\mathcal{R}) = \mathcal{R}$. Direct translocation of quantifiers in (6) yields the following convenient expression for $\rho(\mathcal{S})$, for some $\mathcal{S} \neq \emptyset$, in terms of operations of Minkowski sum and difference:
\[ \rho(\mathcal{S}) = \{ [\mathcal{S} \ominus (-\mathcal{D})] \oplus (-\mathcal{F}) \} \cap \mathcal{S}. \tag{7} \]

In the rest of the paper we will first focus on the characterization of the maximal RCI set $\mathcal{R}_\infty$ for the dynamical system (4) and the generic constraints (5), followed by discussion on explicit form of the maximal RCI set for the particular class of constraint sets. We then proceed to consider a decentralized scenario in which each dynamical system (1), or a collection of systems, implements its own robust control law while guaranteeing robust feasibility for all systems, leading to a concept of decentralized robust control invariance.

**III. CHARACTERIZATION OF RCI SETS FOR DYNAMIC NETWORKS**

Simple dynamics of the system (4) allows us to characterize the corresponding maximal RCI set $\mathcal{R}_\infty$ by a closed–form expression using the formula (6) for $\rho(\cdot)$. We assume the following:

**Assumption 3.1:** (i) $\mathcal{D} \subseteq \mathcal{F}$ and (ii) $\mathcal{X} \cap (-\mathcal{D}) \neq \emptyset$.

Next theorem provides the characterization of the maximal RCI set $\mathcal{R}_\infty$ for the system (4) in terms of the constraints (5).

**Theorem 3.1:** Let the sets $\mathcal{X}, \mathcal{D}$ and $\mathcal{F}$ be bounded. Then, the maximal robust control invariant set $\mathcal{R}_\infty$ for the system (4) subject to constraints (5) is non-empty and is given as:
\[ \mathcal{R}_\infty = \{ [\mathcal{X} \ominus (-\mathcal{D})] \oplus (-\mathcal{F}) \} \cap \mathcal{X}. \tag{8} \]

If and only if the Assumption 3.1 holds.

Theorem 3.1 appears originally in [6], where convexity of the sets $\mathcal{X}, \mathcal{F}$ and $\mathcal{D}$ is assumed and used in the proof. In Appendix A we provide the proof which does not require that the sets $\mathcal{X}, \mathcal{F}$ and $\mathcal{D}$ are convex. Up to this point no assumptions on the constraint set $\mathcal{X}, \mathcal{F}$ and $\mathcal{D}$ are made apart from boundedness. The results in the remainder of the paper will be stated for the particular form of these sets, specified next.

Consider the following bounding functions: $\underline{\mathcal{X}}, \underline{\mathcal{F}}, \underline{\mathcal{D}} : 2^\mathcal{N} \to \mathbb{R}$, where $\underline{\mathcal{X}}(\cdot), \underline{\mathcal{F}}(\cdot), \underline{\mathcal{D}}(\cdot)$ are defined for all $\mathcal{S} \subseteq \mathcal{N}$ and satisfy relations:
\[ \underline{\mathcal{X}}(\mathcal{S}) \leq \overline{\mathcal{X}}(\mathcal{S}), \quad \underline{\mathcal{F}}(\mathcal{S}) \leq \overline{\mathcal{F}}(\mathcal{S}) \leq \overline{\mathcal{D}}(\mathcal{S}) \text{ for all } \mathcal{S} \subseteq \mathcal{N}, \tag{9} \]
with the convention: $\underline{\mathcal{X}}(\emptyset) = \overline{\mathcal{X}}(\emptyset) = \underline{\mathcal{F}}(\emptyset) = \overline{\mathcal{F}}(\emptyset) = \underline{\mathcal{D}}(\emptyset) = \overline{\mathcal{D}}(\emptyset) = 0$. Furthermore, we introduce the capacity function $\overline{\mathcal{E}} : (\mathcal{S}, \mathcal{T}) \to \mathbb{R}^+$, defined for all disjoint subsets $\mathcal{S}$ and $\mathcal{T}$ of the set $\mathcal{N}$, with:
\[ \overline{\mathcal{E}}(\emptyset, \cdot) = \overline{\mathcal{X}}(\emptyset, \cdot) \equiv 0. \]

The constraints on variables $x, \phi, d$ and $u = [u_1 \ldots u_n]^T$ are constructed as:
\[ x \in \mathcal{X} : \underline{\mathcal{X}}(\mathcal{S}) \leq x(\mathcal{S}) \leq \overline{\mathcal{X}}(\mathcal{S}), \forall \mathcal{S} \subseteq \mathcal{N}, \quad \underline{\mathcal{F}}(\mathcal{D}) \leq d(\mathcal{S}) \leq \overline{\mathcal{D}}(\mathcal{S}) \leq \overline{\mathcal{D}}(\mathcal{S}), \forall \mathcal{S} \subseteq \mathcal{N}, \tag{10a} \]
\[ d \in \mathcal{D} := \{ d : \underline{\mathcal{D}}(\mathcal{S}) \leq d(\mathcal{S}) \leq \overline{\mathcal{D}}(\mathcal{S}), \forall \mathcal{S} \subseteq \mathcal{N}, \tag{10b} \]
\[ \phi(\mathcal{S}) = u(\mathcal{S}) - \overline{\mathcal{F}}(\mathcal{S}), \quad \overline{\mathcal{F}}(\mathcal{D}) \leq \overline{\mathcal{F}}(\mathcal{S}), \forall \mathcal{S} \subseteq \mathcal{N} \tag{10c} \]
We remark that the states, inputs and flow constraints defined in (10) represent a wide class of network constraints. For instance, they include the box–type constraints where each variable is individually bounded by some minimal and maximal value. In this case one needs to specify only the functions $g(i), \pi(i), d(i), \tilde{d}(i), u(i), \tilde{u}(i)$, for each node $i \in \mathcal{N}$ and $\pi(i,j)$ for all edges $(i,j) \in \mathcal{E}$ while the remaining combinatorial set of constraints in (10) would be implicitly defined (e.g. $\pi(S) = \sum_{i \in S} \pi(i)$, for all $S \subseteq \mathcal{N}$). General form of the constraints given by (10) is convenient to define aggregated bounds on variables, such as total amount of resource stored in a subset of graph nodes, or the upper bound on the total supply or demand within the network. The particular structure of the set $\mathcal{F}$ postulated by (10c) is motivated by the fact that the set of feasible flows indeed assumes such form, which is immediate consequence of the Gale’s theorem on feasible flows in networks [1].

Note that the sets $\mathcal{X}, \mathcal{U}, \mathcal{D}$ and $\mathcal{F}$ specified by (10) are polytopes. Without loss of generality we impose the following technical assumption on the constraints in (10).

Assumption 3.2: For each non–empty $S \subseteq \mathcal{N}$ there exist $x_s, x_d, d' \in \mathcal{X}$ and $\phi, \tilde{\phi} \in \mathcal{F}$ such that: $w(S) = \pi(S) + \pi(S') + x(S) \geq \min_{y \in \hat{X}} y(S)$, $\forall S \subseteq \mathcal{N}$, (11) where $\hat{X} = \mathcal{X} \cap \{-D\}$.

**Proof:** By Assumption 3.1 and Theorem 3.1, $\mathcal{R}_\infty \neq \emptyset$. Let $x \in \mathcal{R}_\infty$. Then, by Theorem 3.1, $x \in \hat{X}$ and there exists $\phi \in \mathcal{F}$ such that $x + \phi \in \hat{X} = \mathcal{X} \cap \{-D\}$. Since any $y \in \hat{X}$ must satisfy: $\min_{y \in \hat{X}} y(S) \leq y(S) \leq \max_{y \in \hat{X}} y(S)$ for each $S \subseteq \mathcal{N}$, we have:

$$\min_{y \in \hat{X}} y(S) \leq x(S) + \tilde{\phi}(S) \leq \max_{y \in \hat{X}} y(S), \quad (12)$$

Therefore, for this $x \in \mathcal{R}_\infty$, by expression (10c) for the set $\mathcal{F}$:

$$w(S) - \pi(S, S') \leq \tilde{\phi}(S) \leq \max_{y \in \hat{X}} y(S) - x(S),$$

which shows that (a) $\Rightarrow$ (b). The relation (b) $\Rightarrow$ (a) is proved by contradiction: suppose that some $x \in \hat{X}$ satisfies inequalities (11), but $x \notin \mathcal{R}_\infty$ ($\mathcal{R}_\infty \neq \emptyset$ by Assumption 3.1 and Theorem 3.1). Then, for such $x$ there is no $\phi \in \mathcal{F}$ such that $x + \phi \in \hat{X}$, i.e., there is no $\phi \in \mathcal{F}$ such that $\phi \in \hat{X} - x$, or $(\hat{X} - x) \cap \mathcal{F} = \emptyset$. This means, by Assumption 3.2, convexity and closedness of the sets $\hat{X}$ and $\mathcal{F}$, that there exists some $\tilde{S} \subseteq \mathcal{N}$ such that either the hyperplane $\{z: z(S) = \max_{y \in \hat{X}} y(S) - x(S)\}$ separates the polytopes $\mathcal{F}$ and $\mathcal{X} - x$. Assume the former, i.e., that $z(S) \leq \max_{y \in \hat{X}} y(S) - x(S)$ for all $z \in \hat{X} - x$ and $\phi(S) > \max_{y \in \hat{X}} y(S) - x(S)$, for all $\phi \in \mathcal{F}$. By weak redundancy assumption (Assumption 3.2), for $\tilde{S}$ there exists $\phi \in \mathcal{F}$ such that $\phi(S) = \max_{y \in \hat{X}} y(S) - x(S)$ which contradicts the initial assumption that $x$ satisfies inequalities (11). In a similar way the contradiction can also be shown by assuming that $z(S) \geq \min_{y \in \hat{X}} y(S) - x(S)$ for all $z \in \hat{X} - x$ and $\phi(S) = \min_{y \in \hat{X}} y(S) - x(S)$, $\forall \phi \in \mathcal{F}$.

**Remark 3.1:** If the set $\mathcal{X}$ is a box, i.e., $\mathcal{X} = \{x: x_i \leq x_i \leq x_i\}$, then the set $\mathcal{X} = \{-D\}$ is the set of $y \in \mathcal{R}^n$ which satisfy inequalities $\pi(S) + \pi(S') \geq \pi(S) + \pi(S') - x(S), \forall S \subseteq \mathcal{N}$. In this case the set $\mathcal{R}_\infty$ can be explicitly characterized as the set of all $x \in \mathcal{X}$ such that:

$$w(S) - \pi(S, S') + x(S) \leq \sum_{i \in S} \pi(i) + \tilde{d}(i),$$

$$\pi(S) + \pi(S', S) + x(S) \geq \sum_{i \in S} \pi(i) + \tilde{d}(i), \quad \forall S \subseteq \mathcal{N}. \quad (13)$$

**IV. DECENTRALIZED ROBUST CONTROL INVARIANCE**

Implicit characterization of the maximal RCI set for the system (4) and the constraints (10) allows for a simple computation of a feasible flow vector $\phi \in \mathcal{F}$ which, for a given $x \in \mathcal{R}_\infty$, ensures that $x + \phi - d \in \mathcal{R}_\infty$ for all possible realizations of $d$. In general the number of constraints that determines the set of admissible flows $\mathcal{F}$ may grow as $\mathcal{O}(N^2)$, and the centralized robust control of the system (4) may become computationally impractical for a large number of nodes. This motivates our next step, namely to consider a decomposition of the problem into sub–problems through a decentralization. In particular, we decompose the network into disjoint groups of nodes and define a decentralized control law which generates control inputs for each group using only the state information of nodes in the group. Using the general definition of decentralized control law, we show the structure of the resulting decentralized RCI set. Finally, we discuss the problem of selecting the network parameters (capacities and local control bounds) such that the decentralized RCI set is non–empty.

Consider a collection of disjoint sets $\mathcal{D} = \{S_1, \ldots, S_q\}$ such that $\mathcal{N} = \bigcup_{i=1}^q S_i$. We will refer to any such collection as a network decomposition. Without loss of generality we can consider network decompositions that are ordered in the sense that $\min \mathcal{S}_i = \max \mathcal{S}_{i-1} + 1$ for $i \in \mathbb{N}_q$ and that the elements in each $\mathcal{S}_i$ are ordered in increasing order. This is always possible to achieve by re–enumerating the nodes in the graph according to desired association with other nodes. In statements where ordering of nodes is relevant, we will emphasize the fact that we consider ordered network decomposition.

Given a network decomposition $\Delta = \{S_1, \ldots, S_q\}$ the dynamics (4) can be written as:

$$\dot{x}_s^i = x_s^i + u_s^i - \phi_s^i + \phi_{s_i}^+ - d_s^i, \quad i = 1, \ldots, q,$$

where $x_s^i \in X_s^i, u_s^i - \phi_s^i + \phi_{s_i}^+ \in F_{s_i}$, and $d_s^i \in D_s$. For each of the $q$ sub–systems in (14) the implemented control strategy is based on the following decision model.

**Interpretation 1:** At each discrete–time instance, the controller for the $i$th subsystem in (14) has only the information on the current state vector $x_s^i$ and decides on the values of $u_s^i$ and $\phi_{s_i}^+$ at that time instance.

According to Interpretation 1, the information on magnitudes of the inflow $\phi_{s_i}^+$ and the demand $d_s^i$ is not available to the controller implemented for each of the $q$ subsystems. Therefore, $\phi_{s_i}^+$ and the demand $d_s^i$ represent uncertainties in the dynamics of the sub–system comprising the nodes in $S_i$. One could as well select the inflow $\phi_{s_i}^+$ and $u_s^i$ as decision variables for the $i$th sub–system, resulting in no apparent advantage in a general case.

Notions of decentralized control law and decentralized robust control invariant set, based on the decision model specified in Interpretation 1, are defined next.

**Definition 4.1 (Decentralized Control Law):** Control law $\phi(\cdot)$ is decentralized with respect to ordered network decomposition $\Delta$ for
the system (4) and constraints (5) on a domain $\mathcal{P} \subseteq \mathcal{X}$ if: (i) $\phi(x) \in \mathcal{F}$ for all $x \in \mathcal{P}$, and (ii) for any $S_i \in \Delta$ the mappings $u_{S_i}(\cdot)$ and $\phi_{S_i}(\cdot)$ in $\phi_{S_i}(\cdot) = u_{S_i}(\cdot) + \phi_{S_i}(\cdot) - \phi_{S_i}(\cdot)$ are defined on the subspace $\mathbb{R}^{2\mathcal{S}_i}$.

**Definition 4.2 (Decentralized RCI (dRCI) Set):** A set $\mathcal{R}_\Delta$ is a decentralized RCI set for the system (4) subject to constraints (5) with respect to an ordered network decomposition $\Delta$ if there exists a control law $\phi_{S_i}(\cdot)$, decentralized with respect to $\Delta$ on the domain $\mathcal{R}_\Delta$, such that $x + \phi(x) - d \in \mathcal{R}_\Delta$ for any $x \in \mathcal{R}_\Delta$ and for all $d \in \mathcal{D}$.

From the Definition 4.2 one can infer the following structure inherent to any dRCI set.

**Proposition 4.1:** Let $\mathcal{R}_\Delta$ be a decentralized RCI set for the system (4) subject to constraints (5) with respect to ordered network decomposition $\Delta$. Then $\mathcal{R}_\Delta$ is given as a Cartesian product: $\mathcal{R}_\Delta = \prod_{i=1}^{n} \mathcal{R}_{2\mathcal{S}_i}^{\Delta}$, where $\mathcal{R}_{2\mathcal{S}_i}^{\Delta}$ denotes orthogonal projection of the set $\mathcal{R}_{\Delta}$ to the subspace $\mathbb{R}^{2\mathcal{S}_i}$ for $S_i \in \Delta$.

**Proof:** By hypothesis, there exists a decentralized control law $\phi_{S_i}(\cdot)$ defined for all $x \in \mathcal{R}_{\Delta}$. Considering the mapping:

$$\Phi_{\Delta}(x) := \{ \phi \in \mathcal{F} : x + \phi - d \in \mathcal{R}_{\Delta}, \text{ for all } d \in \mathcal{D} \},$$

restricted to the set $\mathcal{R}_{\Delta}$. The set $\Phi_{\Delta}(x)$ is non-empty for all $x \in \mathcal{R}_{\Delta}$ since $\mathcal{R}_{\Delta}$ is an RCI set. Any (point–valued) control law that maps the set $\mathcal{R}_{\Delta}$ onto itself is a selections from the set–valued mapping $\Phi_{\Delta}(\cdot)$. There exists a mapping $\Phi_{\Delta} : \mathcal{R}_{\Delta} \rightarrow 2^\mathcal{F}$ such that $\phi_{S_i}(\cdot)$ is a selection from $\Phi_{\Delta}$ and, since $\phi_{S_i}(\cdot)$ is a decentralized control law (Definition 4.1), the graph of $\Phi_{\Delta}$ is of the form:

$$\text{gph } \Phi_{\Delta} = \left\{ (x, \phi) : x_{S_i} + \phi_{S_i} - \phi_{S_i} \in \mathcal{F}_{S_i}, \ i \in \mathbb{N}_{1, q} \right\},$$

(15)

for some sets $\mathcal{G}_i$ and $\mathcal{F}_i, i \in \mathbb{N}_{1, q}$, and gph $\Phi_{\Delta} \subseteq \text{gph } \Phi_{\Delta}$. The set $\mathcal{R}_{\Delta}$ is obtained by orthogonal projection of the set $\text{gph } \Phi_{\Delta}$ given by (16). Since gph $\Phi_{\Delta}$ is a Cartesian product of sets, so is the set $\mathcal{R}_{\Delta}$.

The property of dRCI sets stated by the Proposition 4.1 reveals the basic principle how to construct dRCI sets, namely to compute RCI sets for individual groups of nodes while treating the flow from the rest of the network as a variable of an unknown value bounded in a certain set. Using that approach we show next how to characterize a particular family of dRCI sets for a network of integrators explicitly parameterized by the bounds that define constraints on state and control variables.

**A. Parametrized family of dRCI sets**

We now impose more structure on the constraints sets $\mathcal{X}, \mathcal{U}, \mathcal{F}$ and $\mathcal{D}$ specified by (10). In particular, we assume that the transfers between the individual nodes $u_{ij}, (i,j) \in \mathcal{E}$, are subject to bounds:

$$u_{ij} \leq u_{ij} \leq \pi_{ij}, \ (i,j) \in \mathcal{E},$$

(17a)

for some finite $u_{ij}$ and $\pi_{ij}, (i,j) \in \mathcal{E}$. Note that constraints (17) fully determine the values of the capacity function $\pi(\cdot)$ used in (10c). We do not restrict the bounds on the transfers $u_{ij}$ between individual nodes $(i,j) \in \mathcal{E}$ to positive or negative values only. Without such restriction, if the control design is centralized, the orientation of edges is superfluous and one could as well consider undirected graph. Note, however, that in the case of decentralized design according to Interpretation 1, the orientation of edges plays a role as it determines which node controls the flow over the edge. According to the convention adopted here, for the edge $(i,j) \in \mathcal{E}$, the node $i$ makes decision on the flow $u_{ij}$. Negative transfer $u_{ij}$ simply means that the node $i$ demands amount $u_{ij}$ from the node $j$ which must satisfy the demand.

The dynamics of the sub-systems in (14) can be compactly written as:

$$x_{S_i}^{k+1} = x_{S_i} + \phi_{S_i} - d_{S_i}, \ i = 1, \ldots, q,$$

(18)

where $\phi_{S_i} := u_{S_i} - \phi_{S_i}$ and $d_{S_i} := d_{S_i} - \phi_{S_i}$. We will now derive explicit form of the constraints on variables $x_{S_i}, \phi_{S_i}$ and $d_{S_i}$.

Let $\nu$ denote the vector of bounds on the transfers $u_{ij}$, i.e., $\nu := [u_{i_1j_1}, \ldots, u_{i_rm}, u_{i_1j_1}, \ldots, u_{i_mn}], (i_k, j_k) \in \mathcal{E}, k \in \mathbb{N}_{1, |\mathcal{E}|}$.

For a non-empty $S \subseteq S_i \in \Delta$ we define:

$$\overline{\sigma}_{S}(\nu) := \sum_{j \in S} \overline{u}_{kj} - \sum_{j \in S} \overline{u}_{jk},$$

(19a)

$$\delta_{S}(\nu) := \sum_{j \in S} \overline{u}_{kj},$$

(19b)

$$\overline{\pi}_{S}(\nu) := \sum_{j \in S} \overline{u}_{kj} - \sum_{j \in S} \overline{u}_{jk},$$

$$\overline{\delta}_{S}(\nu) := \sum_{j \in S} \overline{u}_{kj},$$

(19c)

The terms $\overline{\sigma}_{S}(\nu)$ and $\overline{\pi}_{S}(\nu)$ are, respectively, the lower and upper bound of the component of total net flow $\phi_{S_i}$ transferred to other nodes within $N \setminus S$ and from other nodes in $S$. Similarly, $\delta_{S}(\nu)$ and $\overline{\delta}_{S}(\nu)$ are the bounds on the total inflow from other nodes in $S'$. Using (19) the magnitudes of $\phi_{S_i}$ and $d_{S_i}$ are constrained by the following inequalities:

$$u(S) + \overline{\sigma}_{S}(\nu) \leq \phi_{S_i}(S) \leq u(S) + \overline{\pi}_{S}(\nu),$$

(20a)

$$d(S) - \delta_{S}(\nu) \leq d_{S_i}(S) \leq d(S) - \overline{\delta}_{S}(\nu),$$

(20b)

Introduce scalar parameters $\mu^S, \xi^S, \psi^S, \psi^S$ for each (non-empty) $S \subseteq S_i$ and for each $S_i$ in the given network decomposition $\Delta$. The parameters $\mu^S, \xi^S, \psi^S$ are used to represent lower and upper bounds of $u(S)$ and $x(S)$, respectively. Similarly, introduce the vector that collects parameters representing the bounds on the transfers $u_{ij}$:

$$\pi = [\pi_{i_1j_1}, \ldots, \pi_{i_rm}, \pi_{i_1j_1}, \ldots, \pi_{i_mn}], (i_k, j_k) \in \mathcal{E}, k \in \mathbb{N}_{1, |\mathcal{E}|}.$$

The total number of introduced parameters amounts to $n_{p} = 2|\mathcal{E}| + \sum_{i=1}^{q} |2\mathcal{S}_i| + 1$. Finally, as a notational shorthand, construct the vector $\xi \in \mathbb{R}^{n_{p}}$ that comprises all the above–introduced parameters (for each of the parameters $(\mu^S, \psi^S, \psi^S, S \subseteq S_i \subseteq \Delta$ and each $\pi_{ij}, (i,j) \in \mathcal{E}$, there is exactly one entry in the vector $\xi$). Consider the following mappings:

$$\overline{\pi}^S(\xi) = \left\{ \phi : \phi(S) \leq \mu^S(S) + \overline{\pi}_{S}(\pi), \forall S \subseteq S_i \right\},$$

(21a)

$$\overline{d}^S(\xi) = \left\{ d : d(S) \leq \mu^S(S) + \overline{\delta}_{S}(\pi), \forall S \subseteq S_i \right\},$$

(21b)

$$\overline{\pi}^S(\xi) = \left\{ x : \psi^S \leq x(S) \leq \psi^S, \forall S \subseteq S_i \right\},$$

(21c)

defined on the domain:

$$C = \left\{ \xi \in \mathbb{R}^{n_{p}} : \begin{array}{l}
\mu^S(S) \leq \psi^S \leq \psi^S \leq \overline{\pi}_{S}(\pi), \\
u(S) \leq \mu^S(S) \leq \overline{\pi}_{S}(\pi), \forall S \subseteq S_i, \forall S \subseteq \Delta \\
\overline{u}_{jk} \leq \overline{u}_{jk} \leq \overline{u}_{jk}, (i,j) \in \mathcal{E}
\end{array} \right\}.$$
The expressions in (21) and (22) define explicitly the dependence of the network parameters collected in the argument vector $\xi$ and the constraints on the variables of the dynamical system (18). Given the bounds, specified by functions $\zeta(x)$, $\tau(x)$, $\underline{\pi}(x)$, and $\bar{\pi}(x)$ and the bounds $\underline{y}_j$, $\bar{y}_j$, $(i,j) \in E$, the selection of the parameter vector $\xi \in C$ determines the admissible, non-empty constraint sets $\tilde{F}^S(\xi)$, $D^S(\xi)$ and $\tilde{X}^S(\xi)$ for variables $\phi_{x_i}$, $d_{x_i}$ and $x_{x_i}$ of each of the $q$ dynamical systems (18). Note that for a given $S_i$ the constraints that couple states, inputs and disturbances associated to nodes not belonging to $S_i$, are effectively made redundant or conservatively approximated by inequalities in (22). The price to pay is that, in general, $\bigcup_{\xi \in C} \tilde{F}_i S(\xi) \subseteq \tilde{F}$, $\bigcup_{\xi \in C} \tilde{X}_i S(\xi) \subseteq \tilde{X}$, and $\bigcup_{\xi \in C} D_i S(\xi) \supseteq D$, where the original sets $\tilde{F}$, $\tilde{X}$ and $D$ are given by (10) and (17).

The parametrization of constraints via the mappings in (21) allows us to introduce a family of parametrized decentralized RCI sets for the system (4), decomposed as in (14), as follows.

**Definition 4.3 (Parametrized dRCI set):** Parametrized dRCI set for the system (4) subject to constraints (10) and (17) with respect to an ordered network decomposition $\Delta$ is the set $\tilde{R}^\Delta := \bigcap_{i=1}^q \tilde{R}^{C_i}$ where $\tilde{R}^{C_i}$ is an RCI set for the $i$th sub-system in (14), subject to constraints: $x_{S_i} \in \tilde{X}^{S_i}(\xi)$, $d_{S_i} \in \tilde{D}^{S_i}(\xi)$ and $\bar{d}_{S_i} \in \tilde{D}_i^{S_i}(\xi)$ for some $\xi \in C$, where $\tilde{X}^{S_i}(\xi)$, $\tilde{F}^{S_i}(\xi)$, and $\tilde{D}^{S_i}(\xi)$ are defined in (21). Clearly, any non-empty parametrized dRCI set $\tilde{R}^{C_i}$ for some $\xi \in C$ is a dRCI set according to Definition 4.2. Interesting parameters $\xi \in C$ are those for which the corresponding parametrized dRCI set $\tilde{R}^{C_i}$ is non-empty. From the Definition 4.3 and Theorem 3.1, it is immediate that the set of such parameters is given as:

$$
\Pi := \left\{ \xi \in C : \tilde{F}^S(\xi) \supseteq \tilde{D}_i^S(\xi), \quad \tilde{X}^S(\xi) \supseteq \tilde{D}^S(\xi) \right\} \cup \{ \emptyset \} \quad \forall \xi \in \Delta.
$$

To substantiate our next claim, we need the following lemma:

**Lemma 4.1:** Given a finite $T = \mathbb{N}_{[1,n]}$, let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^n$ be non-empty polytopes of the form:

$$
X := \left\{ x : \begin{bmatrix} \bar{H} & \underline{H} \end{bmatrix} x \leq k \right\}
$$

and

$$
Y := \left\{ y : \begin{bmatrix} \bar{H} & \underline{H} \end{bmatrix} y \leq k \right\},
$$

respectively, where each row of the matrix $H$ corresponds to exactly one $S \subseteq T$. The claim follows directly from the weak redundancy assumption. (ii) Given a polytope $A = \{x : \bar{H} x \leq k\} \subset \mathbb{R}^n$ and a set $B \subset \mathbb{R}^n$, from the definition of the operation $\otimes$ one sees that: $A \otimes B = \{x : h_i x \leq k_i - \max_{a \in B} h_i \}$, where row vector $h_i$ is the $i$th row of the matrix $H$ and $r$ ranges over all row indices of $H$. Applying this to $\mathcal{X} \otimes \mathcal{Y}$ and taking into account assumed weak redundancy, one obtains: $\mathcal{X} \otimes \mathcal{Y} = \{x : \bar{H} x \leq k \}$.

In the view of Lemma 4.1, we can state the following computationally relevant observation on the form of the set of parameters $\Pi$.

**Proposition 4.2:** Consider the dynamical system (4) subject to constraints (10) and (17). Assume that all redundant inequalities in (21) that define the sets $\tilde{F}^S(\xi)$, $D^S(\xi)$ and $\tilde{X}^S(\xi)$ are weakly redundant for all $\xi \in C$. Then the set $\Pi$ given by (23) is a polyhedral set.

The computation of the appropriate vector of parameters $\xi$ for which the parametrized dRCI set is nonempty can be carried out by minimizing some cost function $f(\xi)$, subject to the constraint $\xi \in \Pi$. If the selected parameter cost function $f(\xi)$ is convex, then the problem of synthesis of a decentralized RCI set, according to the Proposition 4.2, can be casted as a convex optimization problem.

In general, for generic type of polytopes (or polyhedra) $\mathcal{F}$, $\mathcal{X}$ and $\mathcal{D}$ that do not have the particular structure as the sets in (10), the conditions that define the set $\Pi$ in (23) cannot be represented by linear inequalities.

We remark that the total number of parameters equals $n_p = 2|\mathcal{E}| + \sum_{i=1}^q 2|\mathcal{S}_i| + 1$ and the number of constraints that define the set $\Pi$ grows as $\sum_{i=1}^q 2|\mathcal{S}_i|$. For $|\mathcal{S}_i|$ much smaller than $|\mathcal{S}|$ this may result in optimization problems of manageable size compared to the centralized solution, both from the computational aspect as well as from the decentralized design and subsequent robust control synthesis.

**B. Decentralized control with communication**

The selection of a suitable parameter $\xi \in \Pi$, equivalent to the selection of the parametrized dRCI set $\tilde{R}^\Delta$, is presented in the previous section as a centralized design process. Once the parameter $\xi$ has been selected, each of the group of nodes is guaranteed to operate without the need to exchange any information. An alternative is a design in which the computation of the parameter $\xi$ is performed at each time instance for a given state $x$, with the constraint that $\xi \in R^\Delta_\xi$. Such constraint is representable by linear inequalities, as evidenced by the Proposition 3.1. The computation of $\xi$ (which is now $\xi(x)$) can be computed by the central authority, using suitable optimality criterion to reflect some preference in the choice of $\xi(x)$, and then passed to each individual group of nodes. The controller associated to each group then selects the control input independently from the other groups, with the ultimate result that $x^+ \in R^\Delta_\xi(x)$. Such hierarchical decomposition is clearly possible for all states $x \in \mathbb{R}^\Delta_\xi$.

One can envisage the scenario in which the nodes belonging to disjoint groups of nodes can communicate and exchange the information relevant for the computation of the parameter $\xi$ (or $\xi(x)$ at each time instance for a given $x$) using distributed implementations of the underlying optimization problem (cf. for instance the monograph [16]).

**V. Numerical Example**

We illustrate the process of selection of parameter $\xi \in \Pi$ on a small example. Consider the network of integrators defined by the graph shown in Figure 1. The upper bounds on the states $\bar{\pi}(i)$, inputs $\underline{\pi}(i)$ and the disturbances $\bar{d}(i)$ are given in Figure 1. Lower bounds on $x_i$ and $d_i$ and $u_i$ are 0 for all $i \in \mathbb{N}_{[1,11]}$. Additional constraints are imposed on the states:

$$
x_1 + x_3 + x_6 \leq 10, \quad x_2 + x_4 + x_5 \leq 10, \quad x_7 + x_8 + x_9 \geq 2.
$$

The considered network decomposition is $\Delta = \{S_1, S_2, S_3, S_4\}$, where $S_1 = \{1, 3, 6\}$, $S_2 = \{2, 4, 5\}$, $S_3 = \{7, 8, 9\}$ and $S_4 = \{10, 11\}$ (see Figure 1). The goal is to find the capacities of the edges connecting the nodes such that the decentralized robust control invariant set exists for the network. The upper and lower bounds $\bar{u}(i)$ and $\underline{\bar{u}}(i)$ are kept fixed, while the edge capacities $\underline{y}_j$ and $\bar{y}_j$, $(i,j) \in E$. 
and Table II is an 11-dimensional polyhedron defined by 360 (non-redundant) inequalities. On the other hand, local sets of feasible flows \( F_{s1}, F_{s2}, F_{s3} \) and \( F_{s4} \) for the computed vector of parameters are defined by 14, 12, 12 and 6 inequalities, respectively.

VI. CONCLUSION

We have shown how insights into specific dynamical properties of networked integrators offer a possibility to introduce a different approach to the problem of constrained decentralized control of such systems. The need for decentralization comes from the complexity issues one faces when dealing with large and complex networks. In our approach to decentralized robust control for a network of integrators the selection of network parameters plays the crucial role in the design. We showed, for a particular type of constraints, how one can compute network parameters for which the robust decentralized control problem has a solution. There are several possible venues of research stemming from this project.

To begin with, a development of a control scheme based on the decentralized framework introduced here, should be addressed. A two level controller, as hinted in Section IV-B, would dynamically select a decentralized RCI set by appropriate selection of its parameters for a given state vector, which would then provide constraints for the second control level where the local control inputs would be computed in a decentralized way.

Of particular interest would be also to investigate a possibility of extending the idea of invariant set parametrization to decentralized control of more general classes of systems and constraints.

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REFERENCES

APPENDIX A

PROOF OF THEOREM 3.1

Recall that the operations Minkowski sum $\oplus$ and Pontryagin difference $\ominus$ can be written as:

$$A \oplus B := \bigcup_{b \in B} A + b, \quad \text{and} \quad A \ominus B := \bigcap_{b \in B} A - b, \quad A, B \subset \mathbb{R}^n;$$

for non-empty $A$ and $B$, where “+” and “−” in the above expression denote translations of the respective sets. Using the above expression and basic operations on sets, the following properties can be verified immediately for non-empty $A, B, C \subset \mathbb{R}^n$:

$$(A \cap B) \ominus C = (A \ominus C) \cap (B \ominus C), \quad \text{(A.1a)}$$

$$(A \oplus B) \ominus C \supseteq A, \quad \text{if} \quad B \supseteq C; \quad \text{(A.1b)}$$

$$(A \ominus B) \ominus C \cap A \neq \emptyset, \quad \text{if} \quad B \subseteq C \text{ and } A \ominus B \neq \emptyset. \quad \text{(A.1c)}$$

We will use operator $\rho(\cdot)$ defined by (6)–(7) as a shorthand notation. First we prove sufficiency of the Assumption 3.1 for the claim. Suppose that Assumption 3.1 holds. Then from (A.1c) it follows that $\rho(X) \neq \emptyset$. We next show that $\rho(X)$ is an RCI set by verifying that $\rho(\rho(X)) = \rho(X)$. We have:

$$\rho(\rho(X)) = \{\rho(X) \ominus (-D) \ominus (-F)\} \cap \rho(X). \quad \text{(A.2)}$$

Note that:

$$\rho(X) \ominus (-D) \ominus (-F) =$$

$$= \left\{ \left[ \left[ \left( X \ominus (-D) \bigoplus (-F) \right) \cap X \right] \ominus (-D) \bigoplus (-F) \right] \cap (-D) \bigoplus (-F) \right\} \cap (-D) \bigoplus (-F) =$$

$$= \left\{ \left[ \left[ \left( [X \ominus (-D)] \bigoplus (-F) \right) \cap [X \ominus (-D)] \bigoplus (-F) \right] \cap X \right] \ominus (-D) \bigoplus (-F) \right\} \cap X =$$

$$= \left[ (X \ominus (-D)) \bigoplus (-F) \right] \cap X = \rho(X).$$

Therefore:

$$\rho(\rho(X)) = \left\{ \left[ \rho(X) \ominus (-D) \bigoplus (-F) \right] \cap \rho(X) =$$

$$= \left[ (X \ominus (-D)) \bigoplus (-F) \right] \cap \rho(X) \right\} \cap \rho(X) =$$

$$= \rho(X) \ominus (-D) \bigoplus (-F) \supseteq \rho(X).$$

The set $\rho(X)$ is an RCI set and, as any RCI set is subset of or equal to $\rho(X)$ by definition of $\rho(X)$, we conclude that $\rho(X) = \mathcal{R}_\infty$.

Next we prove necessity of the Assumption 3.1. Let $\mathcal{R}_\infty = \rho(X) \neq \emptyset$. If $\rho(X) \neq \emptyset$, it is necessary that $X \ominus (-D) \neq \emptyset$ (Assumption 3.1 (ii)). Suppose now that Ass. 3.1(i) does not hold, i.e., that $\mathcal{D} \setminus \mathcal{F} \neq \emptyset$. Since $\rho(X)$ is an RCI set, it is a fixed point of the operator $\rho(\cdot)$:

$$\rho(\rho(X)) = \{\rho(X) \ominus (-D) \bigoplus (-F)\} \cap \rho(X) = \rho(X),$$

from which we conclude that:

$$\rho(X) \ominus (-D) \bigoplus (-F) \supseteq \rho(X),$$

and, adding the set $-D$ to both sides:

$$\rho(X) \ominus (-D) \bigoplus (-D) \bigoplus (-F) \supseteq \rho(X) \ominus (-D). \quad \text{(A.3)}$$

Using the fact that $A \ominus B \bigoplus B \subseteq A$ for any non-empty $A$ and $B$ (easy to verify using the definitions for $\ominus$ and $\oplus$), from (A.3) we obtain:

$$\rho(X) \ominus (-F) \supseteq \rho(X) \ominus (-D) \bigoplus (-D) \bigoplus (-F) \supseteq \rho(X) \ominus (-D).$$

Therefore, we have that: $\rho(X) \ominus (-F) \supseteq \rho(X) \ominus (-D)$. This relation cannot hold for bounded $F$, $D$ and $\rho(X)$ (sets $X$, $F$, $D$ are bounded by hypothesis of Theorem 3.1) if there exists an element of $-D$ which does not belong to $-F$, i.e., if $D \setminus F \neq \emptyset$. Therefore, if $\mathcal{R}_\infty = \rho(X) \neq \emptyset$, then $D \setminus F = \emptyset$, i.e., Assumption 3.1(i) must also hold. This completes the proof.