Reference Governor for Constrained Piecewise Affine Systems

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Abstract

We present a methodology for designing reference tracking controllers for constrained, discrete-time piecewise affine systems. The approach follows the idea of reference governor techniques where the desired set-point is filtered by a system called the “reference governor”. Based on the system current state, set-point, and prescribed constraints, the reference governor computes a new set-point for a low-level controller so that the state and input constraints are satisfied and convergence to the original set-point is guaranteed.

In this note we show how to design a reference governor for constrained piecewise affine systems by using polyhedral invariant sets, reachable sets, multiparametric programming and dynamic programming techniques.

Key words: Model Predictive Control, Process Control

1. Introduction

Different methods for the analysis and design of controllers for hybrid systems have emerged in the past [39, 31, 4]. Among them, the class of
optimal controllers is one of the most studied. The existing approaches differ greatly in the hybrid models adopted, in the formulation of the optimal control problem and in the method used to solve it. In this work we will focus on discrete-time piecewise affine (PWA) models. Discrete-time PWA models can describe a large number of processes, such as: discrete-time linear systems with static piecewise-linearities; discrete-time linear systems with discrete states and inputs; switching systems where the dynamic behavior is described by a finite number of discrete-time linear models together with a set of logic rules for switching among these models; approximation of nonlinear discrete-time dynamics, e.g., via multiple linearizations at different operating points.

This work deals with the design of reference-tracking state-feedback controllers for constrained PWA systems. Our interest stems from industrial practice where, for control synthesis purposes, nonlinear plants are often approximated by partitioning the space spanned by the inputs, state, and exogenous signals into a finite number of regions (also called “modes”). Each region is then assigned an affine model and the nonlinear system is thus approximated by a PWA system. A standard gain scheduling strategy consists of designing a linear controller for each region along with an appropriate strategy for switching between them. It should be noted that the resulting closed-loop system contains reference signals for the controller which are exogenous to the closed-loop. In order to satisfy the state and input constraints, the control designer has to explicitly consider the case where a change in the reference signal results in a system transition between two or more regions.

In principle one could solve an optimal tracking problem for the constrained PWA systems by using the approach presented in [15]. There the authors have characterized the state-feedback solution to optimal control problems for PWA systems with performance criteria based on quadratic and linear norms. They have shown that the solution is a time-varying piecewise affine feedback control law, possibly defined over non-convex regions and proposed an algorithm that solves the Hamilton-Jacobi-Bellman equation by using a simple multiparametric solver. However, the implementation of the explicit controller might require significant computation infrastructure which might not be available on processes with fast sampling time and limited computational resources.

In order to overcome this limitation, one possibility is to design low complexity suboptimal constrained controllers as proposed in [24]. In this paper we present an approach based on the concept of “reference gover-
The idea underlying reference governor is to add a nonlinear device to a controlled system. Such device is called reference governor (RG) and its operation is based on the current state, set-point, and prescribed constraints. Typically the RG selects at any time a virtual reference sequence among a family of linearly parameterized sequences, by solving a convex constrained quadratic optimization problem, and feeds the controlled system according to a receding horizon control philosophy [6]. The overall system is proved to fulfill the constraints, be asymptotically stable, and exhibit an offset-free tracking behavior, provided that an admissibility condition on the initial state is satisfied [6].

This work shows how to design a reference governor for constrained piecewise-affine systems by using polyhedral invariant sets, reachable sets, multiparametric programming and dynamic programming. Compared to the infinite time optimal solution [15, 14] the approach presented in this paper will be less computationally demanding at the price of suboptimality and a smaller region of attraction. In particular our approach can lead to empty or small feasibility regions and it will not succeed if the PWA system cannot be stabilized in any mode (e.g. stability can only be obtained by continuously switching between modes).

We remark that there exist other other very efficient approaches appeared in the literature for solving optimal control problems for PWA systems [25, 1, 11, 38, 18]. The comparison would be problem dependent and requires the simultaneous analysis of several issues such as speed of computation, storage demand and real time code verifiability. This is an involved study and as such is outside of the scope of this paper.

The paper is organized as follows. We give a short overview on multi-parametric programming and on invariant sets in Section 2. The formulation of reference-tracking state-feedback controllers for constrained PWA systems is presented in Section 3. In Section 4 we introduce and discuss the reference governor algorithm. Finally, in Section 5 an example is given that confirms the efficiency of the new method.

2. Definitions and Basic Results

In this section we introduce a few definitions and then recall some basic results on multi-parametric programming and invariant set theory. We will denote the set of all real numbers and positive integers by $\mathbb{R}$ and $\mathbb{N}^+$, respectively.
**Definition 1.** A polyhedron is a set that equals the intersection of a finite number of closed halfspaces.

**Definition 2.** A collection of sets $R_1, \ldots, R_N$ is a partition of a set $\Theta$ if (i) $\bigcup_{i=1}^{N} R_i = \Theta$, (ii) $R_i \cap R_j = \emptyset$, $\forall i \neq j$. Moreover $R_1, \ldots, R_N$ is a polyhedral partition of a polyhedral set $\Theta$ if $R_1, \ldots, R_N$ is a partition of $\Theta$ and the $\bar{R}_i$’s are polyhedral sets, where $\bar{R}_i$ denotes the closure of the set $R_i$.

**Definition 3.** A function $h : \Theta \rightarrow \mathbb{R}^k$, where $\Theta \subseteq \mathbb{R}^s$, is piecewise affine (PWA) if there exists a partition $R_1, \ldots, R_N$ of $\Theta$ and $h(\theta) = H^i \theta + k^i$, $\forall \theta \in R_i$, $i = 1, \ldots, N$.

**Definition 4.** A function $h : \Theta \rightarrow \mathbb{R}^k$, where $\Theta \subseteq \mathbb{R}^s$, is PWA on polyhedra (PPWA) if there exists a polyhedral partition $R_1, \ldots, R_N$ of $\Theta$ and $h(\theta) = H^i \theta + k^i$, $\forall \theta \in R_i$, $i = 1, \ldots, N$.

Piecewise quadratic functions (PWQ) and piecewise quadratic functions on polyhedra (PPWQ) are defined analogously.

### 2.1. Background on Multiparametric programming

Consider the nonlinear mathematical program dependent on a parameter vector $\theta$ appearing in the cost function and in the constraints

$$J^*(\theta) = \inf_z J(z, \theta)$$

$$\text{subj. to } g(z, \theta) \leq 0$$

$$z \in M,$$

where $z \in \mathbb{R}^s$ is the optimization vector, $\theta \in \mathbb{R}^n$ is the parameter vector, $J : \mathbb{R}^s \times \mathbb{R}^n \rightarrow \mathbb{R}$ is the cost function, $g : \mathbb{R}^s \times \mathbb{R}^n \rightarrow \mathbb{R}^m$ are the constraints and $M \subseteq \mathbb{R}^s$. A small perturbation of the parameter $\theta$ in (1) can cause a variety of outcomes, i.e., depending on the properties of the functions $J$ and $g$ the solution $z^*(\theta)$ may vary smoothly or change abruptly as a function of $\theta$. We denote by $K^*$ the set of feasible parameters, i.e.,

$$K^* = \{ \theta \in \mathbb{R}^n | \exists z \in M, g(z, \theta) \leq 0 \},$$

by $R : \mathbb{R}^n \rightarrow 2^{\mathbb{R}^s}$, where $2^{\mathbb{R}^s}$ denotes the set of all subsets of $\mathbb{R}^s$, the point-to-set map that assigns the set of feasible $z$

$$R(\theta) = \{ z \in M | g(z, \theta) \leq 0 \}.$$
to a parameter $\theta$, by $J^* : K^* \to \mathbb{R} \cup \{-\infty\}$ the real-valued function which expresses the dependence on $\theta$ of the minimum value of the objective function over $K^*$, i.e.
\[ J^*(\theta) = \inf_z \{ J(z, \theta) \mid \theta \in K^*, z \in R(\theta) \}, \quad (4) \]
and by $Z^* : K^* \to 2^{\mathbb{R}^s}$ the point-to-set map which expresses the dependence on $\theta$ of the set of optimizers, i.e., $Z^*(\bar{\theta}) = \{ z \in R(\bar{\theta}) \mid J(z, \bar{\theta}) = J^*(\bar{\theta}) \}$ with $\bar{\theta} \in K^*$.

$J^*(\theta)$ will be referred to as the optimal value function or simply value function, $Z^*(\theta)$ will be referred to as the optimal set. We will denote by $z^* : \mathbb{R}^n \to \mathbb{R}^s$ one of the possible single valued functions that can be extracted from $Z^*$, $z^*$ will be called the optimizer function. If $Z^*(\theta)$ is a singleton for all $\theta$, then $z^*(\theta)$ is the only element of $Z^*(\theta)$.

Optimal control problems for nonlinear systems can be reformulated as the mathematical program (1) where $z$ is the input sequence to be optimized and $\theta$ the initial state of the system. Therefore, the study of the properties of $J^*$ and $Z^*$ is fundamental for the study of properties of state-feedback optimal controllers. Fiacco ([19, Chapter 2]) provides conditions under which the solution of nonlinear multiparametric programs (1) is locally well behaved and establishes properties of the solution as a function of the parameters. In this note we restrict our attention to the following special class of multiparametric programming:

\[ J^*(\theta) = \frac{1}{2} \theta' Y \theta + \min_z \frac{1}{2} z' H z + z' F \theta \]
subj. to $C z \leq c + S \theta \quad (5)$

where $z \in \mathbb{R}^s$ is the optimization vector, $\theta \in \mathbb{R}^n$ is the vector of parameters, and $C \in \mathbb{R}^{nc \times s}$, $c \in \mathbb{R}^{nc}$, $S \in \mathbb{R}^{nc \times n}$ are constant matrices. We refer to the problem of computing $z^*(\theta)$ and $J^*(\theta)$ in (5) as (right-hand-side) multiparametric quadratic program (mp-QP).

**Theorem 1** ([5]). Consider the mp-QP (5). Assume $H \succ 0$ and $[Y^T F'] \succeq 0$. The set $K^*$ is a polyhedral set, the value function $J^* : K^* \to \mathbb{R}$ is PPWQ, convex and continuous and the optimizer $z^* : K^* \to \mathbb{R}^s$ is PPWA and continuous.

### 2.2. Background on Invariant Sets

This section adopts the notation used in [23, 37, 27] and provides the basic definitions for invariant sets for constrained systems. A comprehensive survey of papers on set invariance theory can be found in [13].
Denote by $f_a$ the state update function of an autonomous system

$$x(t + 1) = f_a(x(t))$$  \hspace{1cm} (6)

subject to the constraints

$$x(t) \in \mathcal{X}$$  \hspace{1cm} (7)

For the autonomous system (6)-(7), we denote the set of states that evolves to $\mathcal{S}$ in one step as

$$\text{Pre}_{f_a}(\mathcal{S}) \triangleq \{ x \in \mathcal{X} \mid f_a(x) \in \mathcal{S} \}$$  \hspace{1cm} (8)

Equivalently, for the system with inputs

$$x(t + 1) = f(x(t), u(t)),$$  \hspace{1cm} (9)

subject to the constraints

$$x(t) \in \mathcal{X}, \ u(t) \in \mathcal{U},$$  \hspace{1cm} (10)

the set of states which can be driven into the target set $\mathcal{S}$ in one time step is defined as

$$\text{Pre}_f(\mathcal{S}) \triangleq \{ x \in \mathcal{X} \mid \exists u \in \mathcal{U} \text{ s.t. } f(x, u) \in \mathcal{S} \}$$  \hspace{1cm} (11)

Two different types of sets are considered in this note: invariant sets and control invariant sets. We will first discuss invariant sets. The invariant sets are computed for autonomous systems and can be used to “find, for a given feedback controller $u = k(x)$, the set of states whose trajectory will never violate the system constraints”. The following definitions are derived from [13, 10, 9, 28, 20].

**Definition 5 (Positive Invariant Set).** A set $\mathcal{O}$ is said to be a positive invariant set for the autonomous system (6) subject to the constraints in (7), if

$$x(0) \in \mathcal{O} \quad \Rightarrow \quad x(t) \in \mathcal{O}, \quad \forall t \in \mathbb{N}^+$$

Control invariant sets are defined for systems subject to external inputs and can be used to “find the set of states for which there exists a controller such that the system constraints are never violated”. The following definitions are derived from [13, 10, 9, 28].
Definition 6 (Control Invariant Set). A set $C \subseteq X$ is said to be a control invariant set for the system in (9) subject to the constraints in (10), if

$$x(t) \in C \Rightarrow \exists u(t) \in U \text{ such that } f(x(t), u(t)) \in C, \quad \forall t \in \mathbb{N}^+$$

For all states contained in the control invariant set there exists a control law, such that the system constraints are never violated. This does not imply that there exists a control law which can drive the state into a user-specified target set. This issue is addressed in the following by introducing the concept of stabilizable sets.

Definition 7 ($N$-Step Stabilizable Set $K_N(f; O)$). For a given invariant target set $O \subseteq X$, the $N$-step stabilizable set $K_N(f; O)$ of the system (9) subject to the constraints (10) is defined as:

$$K_N(f; O) \triangleq \text{Pref}(K_{N-1}(f; O)), \quad N \in \mathbb{N}^+$$
$$K_0(f; O) = O.$$

From Definition 7, all states $x(0)$ belonging to the $N$-Step Stabilizable Set $K_N(f; O)$ can be driven, through a time-varying control law, to the target set $O$ in $N$ steps and stay in $O$ for all $t \geq N$ while satisfying input and state constraints.

An equivalent definition can be given for the autonomous system (6) subject to the constraints in (7); all states $x(0)$ belonging to the $N$-Step Stabilizable Set $K_N(f_a; O)$ will reach to the target set $O$ in $N$ steps and stay in $O$ for all $t \geq N$ while satisfying state constraints.

3. Problem Formulation

Consider the PWA system

$$x(t+1) = A^i x(t) + B^i u(t) + c^i$$
if $[x(t) u(t)] \in P^i, \quad i = \{1, \ldots, s\}, \quad (12)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $\{P^i\}_{i=1}^s$ is a polyhedral partition of the set of the state and input space $\mathcal{P} \subset \mathbb{R}^{n+m}$. The current index $i$ will be called the system mode, i.e., the PWA system (12) is in mode $i$ at time $t$ if $[x(t) u(t)] \in P^i$.  

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System (12) is subject to hard input and state constraints

\[ x(t) \in \mathcal{X}, \ u(t) \in \mathcal{U} \]

for \( t \geq 0 \), and we denote by *Constrained PWA system (CPWA)* the restriction of the PWA system (12) over the set of states and inputs defined by (13),

\[ x(t + 1) = A^i x(t) + B^i u(t) + c^i \text{ if } \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \tilde{\mathcal{P}}^i, \]

where \( \{\tilde{\mathcal{P}}^i\}_{i=1}^s \) is the new polyhedral partition of the sets of state and input space \( \mathbb{R}^{n+m} \) obtained by intersecting the sets \( \mathcal{P}^i \) in (12) with the polyhedron described by (13). We assume the following.

**Assumption 1.** For a given reference state \( x_{\text{ref}} \) there is a unique input \( u_{\text{ref}} = u_{\text{ref}}(x_{\text{ref}}) \) such that \( x_{\text{ref}} = A^i x_{\text{ref}} + B^i u_{\text{ref}} + c^i \) if \( \begin{bmatrix} x_{\text{ref}} \\ u_{\text{ref}} \end{bmatrix} \in \tilde{\mathcal{P}}^i \).

The function \( u_{\text{ref}}(x_{\text{ref}}) \) is unique either from the properties of system (14) (there is one mode and one \( u_{\text{ref}} \) for each \( x_{\text{ref}} \)) or by construction (i.e., for the given \( x_{\text{ref}} \) the user specifies the desired mode and the corresponding \( u_{\text{ref}} \)).

**Remark 1.** Assumptions 1 is introduced for the sake of simplicity and it is not restrictive. It could be easily removed at the cost of a more complex notation. In particular, if multiple equilibria are allowed, the best \( u_{\text{ref}} \) is typically chosen as a result of an optimization problem [35].

Our objective is to design a state feedback control law \( u(x, x_{\text{ref}}) \) such that the closed loop system

\[ x(t + 1) = A^i x(t) + B^i u(x(t), x_{\text{ref}}) + c^i \text{ if } \begin{bmatrix} x(t) \\ u(x(t), x_{\text{ref}}) \end{bmatrix} \in \tilde{\mathcal{P}}^i, \]

converges to \( x_{\text{ref}} \) and satisfies state and input constraints.

A systematic approach to design constrained reference tracking controllers is to use a receding horizon control policy. We define the following cost function

\[
J_N(U_N, x(0), x_{\text{ref}}) \triangleq \|x_N - x_{\text{ref}}\|^2_P + \sum_{k=0}^{N-1} \left[ \|x_k - x_{\text{ref}}\|^2_Q + \|u_k - u_{\text{ref}}(x_{\text{ref}})\|^2_R \right]
\]
with \( Q = Q' \succeq 0, R = R' \succ 0, P \succeq 0, \| x \|_M^2 = x'Mx \) and consider the constrained finite-time optimal control (CFTOC) problem

\[
J_0^*(x(0), x_{ref}) \triangleq \min_{U_N} J(U_N, x(0), x_{ref}) \quad (17)
\]

subj. to

\[
\begin{align*}
x_{k+1} &= A^i x_k + B^i u_k + c^i \quad \text{if } [x_k^u] \in \tilde{P}^i, \ i = 1, \ldots, s \\
x_{ref,k+1} &= x_{ref,k} \\
k &= 0, \ldots, N - 1 \\
x_N, x_{ref} &\in \tilde{X}_f \\
x_0 = x(0), \ x_{ref,0} = x_{ref}
\end{align*}
\quad (18)
\]

where the column vector \( U_N \triangleq [u'_0, \ldots, u'_{N-1}]' \in \mathbb{R}^{mN} \), is the optimization vector, \( N \) is the optimal control horizon. \( \tilde{X}_f \) is a polyhedral terminal region in the \([x, x_{ref}]\)-space. Note that we distinguish between the input \( u(t) \) and the state \( x(t) \) of plant (14) at time \( t \) and the variables \( u_k \) and \( x_k \) of the optimization problem (18).

We will also denote by \( \tilde{X}_k \subseteq \mathbb{R}^{2n} \) the set of states \( x_k \) and references \( x_{ref} \) that are feasible for (16)-(18):

\[
\tilde{X}_k = \left\{ x \in \mathbb{R}^n, \ x_{ref} \in \mathbb{R}^n \left| \begin{array}{c} \exists u \in \mathbb{R}^m, \\
\exists i \in \{1, \ldots, s\} \\
[x^u] \in \tilde{P}^i \ \text{and} \\
[A^i x + B^i u + c^i, x_{ref}] \in \tilde{X}_{k+1} \end{array} \right. \right\}, \quad k = 0, \ldots, N - 1, \\
\tilde{X}_N = \tilde{X}_f.
\quad (19)
\]

Note that the optimizer function \( U_N^* \) may not be uniquely defined if the optimal set of problem (16)-(18) is not a singleton for some \( x(0) \). The next theorem shows the properties of the optimal control solution.

**Theorem 2 ([15]).** Consider the optimal control problem (16)-(18). Then, there exists a solution in the form of a PWA state-feedback control law

\[
u_k^*(x(k), x_{ref}) = F_k^x x(k) + F_k^u u_{ref} + F_k^r x_{ref} + G_k^i \quad \text{if } [x(k), x_{ref}] \in \mathcal{R}_k^i,
\quad (20)
\]

where \( \mathcal{R}_k^i, \ i = 1, \ldots, N_k \) is a partition of the set \( \tilde{X}_k \) of feasible states \( x(k) \) and reference \( x_{ref} \). The boundaries of the sets \( \mathcal{R}_k^i \) are linear and quadratic inequalities in \( x(k) \) and \( x_{ref} \).
Note that quadratic inequalities in the sets $\mathcal{R}_k^i$ arise from the comparison of quadratic costs associated to admissible switching sequences [15].

An infinite horizon controller can be obtained by implementing in a receding horizon fashion a finite-time optimal control law. In this case the control law is simply obtained by repeatedly evaluating at each time $t$ the PWA controller (20) for $k = 0$:

$$u(t) = u^*_0(x(t), x_{ref}) \text{ for } \begin{bmatrix} x(t) \\ x_{ref} \end{bmatrix} \in \tilde{X}_0.$$

If $\tilde{X}_f$ is a control invariant set and the terminal cost $P$ is a control Lyapunov function, then for all $[x(0), x_{ref}] \in \tilde{X}_0$ the system state $x(k)$ will converge to the desired constant reference $x_{ref}$ while satisfying input and state constraints [33]. Note that the sets $\tilde{X}_k$ are defined in the state and reference space, since the initial state and the reference are both external parameters of the optimal control problem (16)-(18).

The number of regions in the solution to (21) might prohibit the real-time implementation for systems with limited computational and storage resources. In the next section we propose an alternative approach based on the results presented in [6, 7, 8, 3, 2] and show how to design a low-complexity controller which guarantees constraint satisfaction by using the idea of reference governor.

4. Reference Governor

Consider the constrained PWA system (14). The proposed control design approach is based on the following three main steps. First, local tracking controllers are designed for each mode $i$ of the PWA system and the invariant sets $\mathcal{O}^i$, in the state and reference space, are computed for the corresponding closed loop systems. Second, for any pair of modes $(i, j)$, transition controllers are designed for steering the current state in mode $i$ to the invariant set $\mathcal{O}^j$ in mode $j$. Last, for any pair of modes $(i, j)$, an optimal sequence of transitions is computed from the mode $i$ to the mode $j$ as the shortest path on a weighted graph. The graph weights are functions of the “transition cost” between any two modes. The online reference governor algorithm solves a simple constrained Quadratic Programming (QP) problem in order to modify the reference and move to the next mode according to the determined shortest path. The three steps are detailed next.
1. **Computation of local tracking controllers**

For each mode \(i\) design a state-feedback controller \(k^i(x, x_{ref})\) for tracking the reference \(x_{ref}\) in mode \(i\). Denote by \(O^i\) the positive invariant set of the corresponding closed loop system. \(O^i\) is a polyhedron in the \([x, x_{ref}]\) space such that if at time \(k\) \([x(k), x_{ref}] \in O^i\), then state and input constraints (13) are satisfied for all \(t \geq k\) and \(x(t) \rightarrow x_{ref}\).

2. **Computation of transition controllers**

For each pair \(i, j\) of modes, design a transition controller \(k^{ij}(x, x_{ref})\). Then, for the closed loop system \(f^{ij}_{a}\), compute the corresponding \(N^{ij}\)-step stabilizable set \(X^{ij} = \mathcal{K}_{N^{ij}}(f^{ij}_{a}, O^j)\), i.e., the set of states and references in mode \(i\) which are steered by \(k^{ij}(x, x_{ref})\) to the invariant set \(O^j\) in mode \(j\) in at most \(N^{ij}\) steps. \(X^{ij}\) is the union of polyhedra in the \((x, x_{ref})\) space. If at time \(k\) there exists a \(\bar{x}_{ref}(k)\) such that \([x(k), \bar{x}_{ref}(k)] \in X^{ij}\), then there exists \(p \leq N^{ij}\) and a reference trajectory \(\bar{x}_{ref}(k), \ldots, \bar{x}_{ref}(k+p)\) such that \([x(k+p), \bar{x}_{ref}(k+p)] \in O^j\) under the control law \(k^{ij}(x(k), \bar{x}_{ref}(k)), \ldots, k^{ij}(x(k+p-1), \bar{x}_{ref}(k+p-1))\). Denote by \(X^{ij}_x\) the projection of \(X^{ij}\) on the \(x\) space, i.e., \(X^{ij}_x = \text{Proj}_x(X^{ij})\). If \(x\) belongs to \(X^{ij}_x\) then there exists a (time-varying) reference which steers the PWA system (12) from mode \(i\) to the invariant in mode \(j\) while satisfying state and input constraints.

3. **Computation of optimal switching policy**

For each pair \(i, j\) of modes, compute the best sequence of transitions \(\{i, i_1, i_2, \ldots, i_p, j\}\) from mode \(i\) to mode \(j\) as the shortest path on a weighted graph. The nodes of the graph represent the system modes and the weights on the arcs represent the cost of switching between two adjacent modes.

The three steps outlined above are discussed in detail in the next Sections 4.1, 4.2, 4.3, respectively. Once the controllers and the invariant sets have been computed, an online reference governor ensures that the system converges to the desired reference while satisfying input and state constraints. Next we provide a simplified description of the on-line reference governor algorithm which will help the reader to understand the main idea. The algorithm will be detailed later in Section 4.4.
Algorithm 4.1.

Input: Current state $x$ and reference $x_{ref}$
Output: Modified reference $\bar{x}_{ref}^*$ and control input $u(x, \bar{x}_{ref}^*)$

1. Read $x$ and $x_{ref}$ and let $r$ be the mode of the reference: $[x_{ref}, u_{ref}(x_{ref})] \in \bar{P}^r$.
2. if $[x_{ref}, x_{ref}] \notin O^r$, $x_{ref}$ is an infeasible reference, EXIT.
3. else if $[x, x_{ref}] \in O^r$ then set $\bar{x}_{ref}^* = x_{ref}$ and apply $u = k^r(x, \bar{x}_{ref}^*)$.
4. else if $\bar{x}_{ref}^*$ exists such that $[x, \bar{x}_{ref}^*] \in O^r$ then choose $\bar{x}_{ref}^*$ with $[x, \bar{x}_{ref}^*] \in O^r$ “close to” $x_{ref}$ and apply $u = k^r(x, \bar{x}_{ref}^*)$.
5. else Let $V = \{v_1, \ldots, v_p\}$ be the set of modes to which $x$ can be steered to, i.e., $\exists j \in \{1, \ldots, n\}, \exists v_k \in V$ such that $x \in X^j_{v_k}$. Note that $j$ might depend on the applied input.
6. Compute the mode $v^*$ belonging to $V$ with associated minimum cost to reach $r$. If there exists no path from any $v_k \in V$ to $r$, then the problem is infeasible, EXIT.
7. Compute $\bar{x}_{ref}^*$ such that $[x, \bar{x}_{ref}^*] \in X^j_{v^*}$ and apply $u = k^j,v^*(x, \bar{x}_{ref}^*)$.
9. go to Step 1.

Note that Assumption 1 has been used in Step 1 of the Algorithm where it is assumed that $u_{ref}(x_{ref})$ is unique.

4.1. Local Control design

For each region $\bar{P}^i$, the following reference tracking controller is considered

$$u = k^i(x, x_{ref})$$

where $k^i(x, x_{ref})$ is a linear control law or a PWA control law. For each region $\bar{P}^i$ we compute a positive invariant set $O^i$ for the closed loop system:

$$x_{k+1} = A^i x_k + B^i k^i(x_k, x_{ref,k}) + c^i,$$
$$x_{ref,k+1} = x_{ref,k},$$

subject to the constraints

$$[x_k, x_{ref,k}] \in \bar{P}^i, \quad [x_{ref,k}, u_{ref}(x_{ref,k})] \in \bar{P}^i.$$
We remark that $\mathcal{O}^i$ is a set in the $[x, x_{ref}]$ space. We assume that $k^i$ guarantees the convergence of $x(k)$ to a constant reference $x_{ref}$ for system (23).

In addition to standard linear control design techniques, the controller $k^i(x, x_{ref})$ can be designed as a receding horizon controller. Consider the following optimal control problem in mode $i$ (denoted as “Problem $i$”) with horizon $N_i$.

$$J_0^*(x(0), x_{ref}) \triangleq \min_{U_{N_i}} J(U_{N_i}, x(0), x_{ref})$$

subject to

$$
\begin{align*}
  x_{k+1} &= A^i x_k + B^i u_k + c^i \\
  [x_k] &\in \tilde{\mathcal{P}}^i, \quad [x_{ref}] \in \tilde{\mathcal{P}}^i \\
  k &= 0, \ldots, N_i - 1 \\
  [x_{N_i}, x_{ref}] &\in \tilde{X}_f^i \\
  x_0 &= x(0).
\end{align*}
$$

(25)

Denote by $\mathcal{X}_0^i$ the feasible set of initial conditions for Problem $i$ (25) and the associated PWA RHC control law

$$k^i(x, x_{ref}) = u_0^*(x, x_{ref}) \text{ for } x \in \mathcal{X}_0^i. \quad (26)$$

If persistent feasibility and convergence are ensured, then $\mathcal{X}_0^i$ is a positive invariant set for system (23)-(24) and $x(k) \to x_{ref}$ and we set $\mathcal{O}^i = \mathcal{X}_0^i$.

We refer the reader to [16] for a discussion on the properties of Problem $i$ guaranteeing that $x(k) \to x_{ref}$ for $k \to \infty$.

**Remark 2.** Note that in problem (16)-(18) the terminal set $\tilde{X}_f$ is an invariant set for the PWA system (10). In problem (25) $\tilde{X}_f$ is a “local” invariant set, i.e., an invariant in mode $i$. $\tilde{X}_f$ is empty if $[x_{ref}, u_{ref}(x_{ref})] \notin \tilde{\mathcal{P}}^i$.

Assume $[x_{ref}, u_{ref}] \in \tilde{\mathcal{P}}^i$. If $[x_{ref}] \in \mathcal{O}^i$ then the controller $k^i(x, x_{ref})$ will (i) guarantee constraint satisfaction at all time instants, (ii) keep the system in mode $i$ and (iii) guarantee convergence to $[x_{ref}, u_{ref}]$ (step 3 of the Online Algorithm). If $[x_{ref}] \notin \mathcal{O}^i$ then the local controller $k^i$ will not guarantee feasibility and will not drive $x_0$ towards $x_{ref}$. However, two cases are possible:

1. a $\bar{x}_{ref}$ might exist such that $[x_0, \bar{x}_{ref}] \in \mathcal{O}^i$.
2. a $\bar{u}$ such that $[x_0] \in \mathcal{P}^l$ and a “transition controller” $k^{l,i}(x, \bar{x}_{ref})$ that steers the system from mode $l$ to mode $i$ through a modified $\bar{x}_{ref}$.

In the both cases feasibility can be guaranteed by computing a new reference $\bar{x}_{ref}$. The second case requires “transition controllers”. The design of such controllers is described next.
4.2. Transition Control Design

For each \((i, j)\), \(i \neq j\), select an horizon \(N_{i,j}\). For a given linear or PWA transition controller \(k_{i,j}^a(x, x_{ref})\), denote by \(f_{i,j}^a\) the closed loop PWA system in region \(i\), i.e., \([x_{k+1}, x_{ref,k+1}] = f_{i,j}^a(x_k, x_{ref,k}) \triangleq [A^i x_k + B^i k_{i,j}^a(x_k, x_{ref,k}) + c^i]\) and by \(X_{i,j}\) the set of states and references which are steered from mode \(i\) to the set \(O_j\) in mode \(j\) in at most \(N_{i,j}\) steps, i.e., \(X_{i,j} \triangleq K_{N_{i,j}}(f_{i,j}^a, O_j)\). Note that if \([x_0, x_{ref}] \notin X_{i,j}\), then the reference \(x_{ref}\) can be modified to \(\bar{x}_{ref}\) in order to have \([x_0, \bar{x}_{ref}] \in X_{i,j}\) and steer the system to mode \(j\) by using the controller \(k_{i,j}^a\). Clearly \(X_{i,j}\) might be empty.

In addition to standard linear control design techniques, \(k_{i,j}^a\) can be designed as a constrained minimum time controller as in [23, 26, 12, 32]. Consider the minimum time control problem

\[
J^*_{0,T}(x(0), x_{ref}) \triangleq \min_{U_T} J(U_T, x(0), x_{ref})
\]

\[
\text{subj. to } \begin{cases} x_{k+1} = A^i x_k + B^i u_k + c^i \\
[x_k] \in \tilde{P}^i, \ k = 0, \ldots, T - 1 \\
[x_0, x_{ref}] \in O_j \\
x_0 = x(0)
\end{cases}
\]

and solve it for \(T = 0, \ldots, N_{i,j} - 1\). This can be done by using a sequence of multi-parametric programs of prediction horizon 1 as proposed in [23]:

\[
J^*_{p+1,N_{i,j}}(x(p+1), x_{ref}) \triangleq \min_{u(p)} J(u(p), x(p), x_{ref})
\]

\[
+ J^*_{p+1,N_{i,j}}(x(p+1), x_{ref})
\]

\[
\text{subj. to } \begin{cases} x_{p+1} = A^i x_p + B^i u_p + c^i \\
[x_p] \in \tilde{P}^i, \ k = 0, \ldots, p - 1 \\
[x_p, x_{ref}] \in X_{p+1}^{i,j}
\end{cases}
\]

for \(p = N_{i,j} - 1, N_{i,j} - 2, \ldots, 0\) with

\[
X_{N_{i,j}}^{i,j} = O_j
\]

and \(X_{p}^{i,j}\) being the set of feasible states \(x(p)\) and references \(x_{ref}\) for which (29) is feasible at time \(p\). Therefore \(N_{i,j}\) multi-parametric programs are solved yielding \(u_{p}^{i,j}\) and \(X_{p}^{i,j}\) for \(p = N_{i,j} - 1, N_{i,j} - 2, \ldots, 0\). Therefore

\[
X^{i,j} = \bigcup_{p=1}^{N_{i,j}} X_{p}^{i,j}
\]
Note that since $N^{i,j}$ multi-parametric programs are solved, several controller regions in $\mathcal{X}^{i,j}_p$ may overlap. In order to guarantee minimum-time convergence and feasibility, the feedback law $u^r_{i,j}$ associated with the region computed at the smallest horizon $c$ is selected for any given state $x$. More details can be found in [23].

Recall that $\mathcal{X}^{i,j}_x$ is the projection of $\mathcal{X}^{i,j}$ on the $x$ space. Similarly, $\mathcal{X}^{i,j}_{x,p}$ is the projection of $\mathcal{X}^{i,j}_p$ on the $x$ space. If $x$ belongs to $\mathcal{X}^{i,j}_{x,p}$ then there exist a reference which will bring the PWA system from mode $i$ to the invariant set $\mathcal{O}^j$ in mode $j$ in $p$ steps.

4.3. The Weighted Graph

For each mode we have designed a local controller $k^i$ and computed a corresponding invariant $\mathcal{O}^i$. For each pair of modes we have designed a transition controller $k^{i,j}$ and computed a set $\mathcal{X}^{i,j}$ of states and references in mode $i$ which reach $\mathcal{O}^j$ in mode $j$ in at most $N^{i,j}$ steps.

Clearly, if the current state is in mode $i_1$ and the reference in mode $i_n$, the system could still be controlled to the reference even if $\mathcal{X}^{i_1,i_n}$ is empty. Therefore, the last step is to compute the optimal transition sequence $i_1, i_2, \ldots, i_n$ between any two modes $i_1$ and $i_n$. We propose to use the properties of the sets $\mathcal{O}^i$ and $\mathcal{X}^{i,j}$ in order to avoid the inherent exponential complexity of the problem at the price of smaller regions of attraction.

In particular we can move from $i_1$ to $i_2$ through the set $\mathcal{X}^{i_1,i_2}$ by using $k^{i_1,i_2}$. Then, when the system is in mode $i_2$, we can move through the set $\mathcal{O}^i_2$ by using $k^{i_2}$ and reach $\mathcal{X}^{i_2,i_3}$ and so on. In this way input and state constraints are always satisfied. The feasibility property of this approach is described in following proposition.

**Proposition 1.** Let Assumption 1 hold, $x(k)$ be the current system state and $x_{ref}$ the current reference. Assume that $[x_{ref}, u_{ref}(x_{ref})]$ is in mode $j$. Assume that $x(k) \in \text{Proj}_x \mathcal{O}^i$. Define the set $\tilde{\mathcal{X}}^{i,j}$ as

$$\tilde{\mathcal{X}}^{i,j} \triangleq \text{Proj}_x \mathcal{O}^i \cap \text{Proj}_x \{[x, x_{ref}] \in \mathcal{X}^{i,j} | x = x_{ref}\} \quad (31)$$

If $\tilde{\mathcal{X}}^{i,j}$ is not empty, then there exists a time varying reference and a feasible state feedback control law such that the system (14) with initial state $x(k)$ in mode $i$ can be steered to the set $\mathcal{O}^j$ in mode $j$.

**Proof.**
We consider two cases: $x(k) \in \text{Proj}_x(X_{i,j})$ and $x(k) \notin \text{Proj}_x(X_{i,j})$.

If $x(k) \in \text{Proj}_x(X_{i,j})$ then compute $\bar{x}_{ref}$ such that $[x(k), \bar{x}_{ref}] \in X_{i,j}$ and apply $k^{i,j}(x(k), \bar{x}_{ref})$. By construction of $X_{i,j}$, there exists a sequence of references such that the state will reach $O^j$ in at most $N_{i,j}$ steps.

Assume $x(k) \notin \text{Proj}_x(X_{i,j})$. By assumption $x(k) \in \text{Proj}_x(O^i)$, and $X_{i,j}$ is not empty. Therefore the system can say in $O^i$ (and thus satisfy constraints) and by changing the reference it can reach a new state $\hat{x}_{ref}$ with the property $\hat{x}_{ref} \in \text{Proj}_x(X_{i,j})$.

Pick $[\hat{x}_{ref}, \tilde{x}_{ref}] \in X_{i,j}$ with $\hat{x}_{ref} \in \bar{X}_{i,j}$ and solve the following problem

$$\bar{x}_{ref} \triangleq \arg \min_{\tilde{x}_{ref}} \|\bar{x}_{ref} - \tilde{x}_{ref}\|$$

subj. to $[x(k), \bar{x}_{ref}] \in O^i$ (32b)

Problem (32) is feasible since by assumption $x(k) \in \text{Proj}_x(O^i)$. Note however that $[x(k), \bar{x}_{ref}]$ might not belong to $O^i$. Apply $k^{i,j}(x, \bar{x}_{ref})$. Since $(x(k), \bar{x}_{ref}) \in O^i$ then $\lim_{k \to +\infty} x(k) = \bar{x}_{ref}$ and since $O^i$ is a connected set, $\lim_{k \to +\infty} \tilde{x}_{ref} \rightarrow \hat{x}_{ref}$ with $[\hat{x}_{ref}, \bar{x}_{ref}] \in X_{i,j}$.

Proposition 1 shows that we can transition from mode $i$ to mode $j$ from $X_{i,j}$ or from $O^i$ if $\bar{X}_{i,j}$ is not empty. Therefore one can steer the state from mode $i_1$ to mode $i_n$ by applying the sequence of controllers $k^{i_1,i_2}, k^{i_2,i_3}, k^{i_3}, \ldots, k^{i_n}$.

The concept of weighted graph will be used to compute the “best” transition sequence from $O^i$ to $O^j$ for any two modes $i, j$. A weighted graph $\mathcal{G}$ is defined as

$$\mathcal{G} = (\mathcal{V}, \mathcal{A})$$

where $\mathcal{V}$ is the set of nodes (or vertices) $\mathcal{V} = \{1, \ldots, N\}$ and $\mathcal{A} \subseteq \mathcal{V} \times \mathcal{V}$ the set of edges $(i, j)$ with $i \in \mathcal{V}, j \in \mathcal{V}$. Let $A_{i,j} \in \mathbb{R}$ be the $i, j$ element of the weighted adjacency matrix $A$ of the graph $\mathcal{G}$. If there is no edge connecting the vertex $i$ with the vertex $j$, i.e., $(i, j) \notin \mathcal{A}$, we set $A_{i,j} = 0$.

The elements of $A$ are computed as follows:

$$a_{i,j} = \alpha \frac{1}{\text{vol}(X_{i,j})} + \beta N_{i,j}$$

where $\text{vol}(P)$ is the volume of the polyhedron $P$. The positive real numbers $\alpha$ and $\beta$ are tuning parameters. Given the weighted graph $\mathcal{G}$, $u =$
SPath(\(G, i_1, i_n\)) is the vector which describes the shortest path

\[ u = \{i_1, i_2, \ldots, i_n\} \]

between node \(i_1\) and node \(i_n\) and SPathCost(\(G, i_1, i_n\)) is the corresponding optimal cost.

Clearly, several other alternatives to the weight choice in (34) can be proposed depending on the specific application. The weights in (34) capture only two important elements: the time to reach the target region and the size of the feasible set which generates a transition. The latter can be seen as a practical measure on how robust to system uncertainties and measurement noise the transition is.

4.4. On-line Reference Governor Algorithm

Once all the elements have been computed off-line, the following algorithm is implemented on-line.
Algorithm 4.2.

Input: Current state \( x(t) \) and reference \( x_{\text{ref}} = x_{\text{ref}}(t) \)
Output: Modified reference \( \bar{x}_{\text{ref}}(t) \) and controller selection

1. Let \( r \) be such that \([x_{\text{ref}}, x_{\text{ref}}] \in O^r\)
2. If \( x(t) \in \text{Proj}_x(O^r) \) then select local controller \( k^r \) and compute the modified reference as follows

   \[ \bar{x}_{\text{ref}} = \arg\min_{\bar{x}_{\text{ref}}} \|\bar{x}_{\text{ref}} - x_{\text{ref}}\| \quad (35a) \]
   \[ \text{subj. to } \begin{bmatrix} x(t) \\ \bar{x}_{\text{ref}} \end{bmatrix} \in O^r \quad (35b) \]

3. Else
4. Let \( v = \{v_1, \ldots, v_n\} \) the set of modes such that \( x(t) \in \text{Proj}_x(\mathcal{X}^{l,v}) \) and let \( u = \{u_1, \ldots, u_m\} \) the set of modes such that \( x(t) \in \text{Proj}_x(\mathcal{O}^{u}) \). (note that \( x(t) \) can be in multiple modes because the system partition depends on the input as well).
5. Compute \( v^* \in v \cup u \) with the associated shortest path \( \{v^*, i_1, \ldots, i_n, r\} \) and cost \( s^* = \text{SPathCost}(G, v^*, r) \).
6. If \( s^* = \infty \) then “Infeasible Reference”, EXIT
7. If \( v^* \in v \) then select transition controller \( k^{l,v^*} \) and compute the modified reference as follows

   \[ \bar{x}_{\text{ref}} = \arg\min_{\bar{x}_{\text{ref}}} \|\bar{x}_{\text{ref}} - x_{\text{ref}}\| \quad (36) \]
   \[ \text{subj. to } \begin{bmatrix} x(t) \\ \bar{x}_{\text{ref}} \end{bmatrix} \in \mathcal{X}^{l,v^*} \quad (37) \]

8. Else select local controller \( k^{v^*} \) and compute the modified reference as follows

   \[ \bar{x}_{\text{ref}} = \arg\min_{\bar{x}_{\text{ref}}},\hat{x}_{\text{ref}}(\|\bar{x}_{\text{ref}} - \hat{x}_{\text{ref}}\|) \quad (38) \]
   \[ \text{subj. to } \begin{bmatrix} x(t) \\ \bar{x}_{\text{ref}} \end{bmatrix} \in \mathcal{O}^{v^*} \quad (39) \]
   \[ \begin{bmatrix} \hat{x}_{\text{ref}} \\ \hat{x}_{\text{ref}} \end{bmatrix} \in \mathcal{X}^{v^*,i_1} \quad (40) \]
9. Go to Step 1

Remark 3. Note that the sets \( \mathcal{X}^{i,j} \) might be described as the union of poly-
hedra $X^{i,j}_k$, for $k = 1, \ldots, N^{i,j}$ where $X^{i,j}_k$ represents the $k$-steps reachable set. In this case, Step 7 in Algorithm 4.2 can be modified as follows:

$$
\bar{x}_{ref}^* = \arg \min_{\bar{x}_{ref,k}} \|\bar{x}_{ref} - \bar{x}_{ref}\| \quad (41a)
$$
$$
\text{subj. to } \left[ \begin{array}{c} \bar{x}(t) \\ \bar{x}_{ref} \end{array} \right] \in X^{i,j}_k \quad (41b)
$$

The same modification can be applied to Step 8 in Algorithm 4.2.

**Remark 4.** Note that the QP problem defined in Step 2 in Algorithm 4.2 can be solved explicitly as has been shown in [36, 34].

**Remark 5.** Consider step 5 of Algorithm 4.2. If there exists multiple $v^*$ yielding $s^* = SPathCost(G, v^*, r)$, then a $v^*$ will be used. In this case cycling could occur but it can easily be avoided by storing modes which have been already explored.

5. Numerical Example

A simple vehicle dynamics example is presented next. We consider a bicycle model of a vehicle sketched in Figure 1 and refer the reader to [17] for details on modeling and simplifying assumptions.

The lateral and yaw dynamics are described by the following nonlinear differential equations:

$$
m\ddot{y} = -m\dot{x}\dot{\psi} + 2 \left[ F_{yf} + F_{yr} \right] \quad (42a)
$$

$$
I\ddot{\psi} = 2 \left[ aF_{yf} - bF_{yr} \right] \quad (42b)
$$

where $\dot{x}$ is the longitudinal speed, $\dot{y}$ the lateral speed, $\dot{\psi}$ the yaw rate, $F_{yf}$ and $F_{yr}$ are front and rear tire lateral forces, respectively.

The front and rear tire slip angles are approximated as:

$$
\alpha_f = \delta_f - \frac{\dot{y} + a\dot{\psi}}{\dot{x}} \quad (43a)
$$

$$
\alpha_r = \frac{b\dot{\psi} - \dot{y}}{\dot{x}} \quad (43b)
$$

where $\delta_f$ is the vehicle steering angle and it is the system input.
The front and rear lateral forces $F_{yf}$ and $F_{yr}$ are approximated as piecewise linear functions of the tire slip angles:

$$F_y(\alpha) = \begin{cases} 
- C_{lin}^{\text{c}} \alpha^* + C_{sat}^{\text{c}} (\alpha + \alpha^*), & \text{for } \alpha < -\alpha^* \\
C_{lin}^{\text{c}} \alpha, & \text{for } -\alpha^* \leq \alpha \leq \alpha^* \\
C_{lin}^{\text{c}} \alpha^* + C_{sat}^{\text{c}} (\alpha - \alpha^*), & \text{for } \alpha > \alpha^* 
\end{cases}$$

(44)

where $C_{lin}^{\text{c}}$ and $C_{sat}^{\text{c}}$ are the slopes of the lateral tire force characteristic in the intervals $[-\alpha^*, \alpha^*]$ and $[-\infty, -\alpha^*] \cup [\alpha^*, \infty]$, respectively. Note that equation (44) has to be considered at both front and rear axis. The system is subject to the following state and input constraints:

$$-10 \leq \dot{\psi} \leq 10, \quad -0.43 \leq \dot{\psi} \leq 0.43, \quad -0.17 \leq \delta_f \leq 0.17$$

The system has nine modes of operation, the combination of three modes at the front axis and three at the rear. Four modes are infeasible and thus only five modes are relevant. We denote as mode 1 the mode where $\alpha_f \in [-\alpha^*, \alpha^*]$ and $\alpha_r \in [-\alpha^*, \alpha^*]$.

We have implemented the RG controller presented in Section 4 and a 2-norm minimum time controller for the resulting PWA systems designed by using the Multiparametric Programing Toolbox (MPT) [30]. The 2-norm minimum time controller with max horizon of ten steps consists of 318 regions. The RG controller consists of five local linear tracking controllers
$k^i(x, x_{ref})$ designed by using LQR techniques. We used the local controllers as transition controllers, i.e., $k^{i,j}(x, x_{ref}) = k^i(x, x_{ref})$ for all $j = 1, \ldots, 5$. With this choice, the only non-empty transition sets reach mode 1. Therefore the RG controller requires five local invariant sets $O_1^\infty, \ldots, O_5^\infty$, and ten transition sets $X_{2,1}^1, X_{3,1}^2, X_{3,1}^3, X_{3,1}^4, X_{1,1}^2, X_{1,1}^5, X_{2,1}^5$. We remark that transition sets with longer horizons are empty.

Figure 2 depicts the feasible sets for both controllers. As expected, the price of complexity reduction is paid with a smaller feasibility region. We remark that in both cases the feasible area is also the region of attraction for the closed loop system.

![Figure 2: Comparison of feasible domains for the RG controller and an explicit minimum-time controller](image)

Next we consider a transition between only two operating modes of the system. In particular we consider mode 1 and mode 2, where $\alpha_f \in [-\alpha^*, \alpha^*]$ and $\alpha_r > \alpha^*$. Our control objective is to steer the system from mode 2 to mode 1. This corresponds to a well known vehicle stability control problem. In particular, mode 2 corresponds to an operating condition denoted as “oversteering” where the rear tire forces saturate. Mode 1 corresponds to a neutral or “stable” driving condition. Figure 3 reports a simulation in the phase plane where the vehicle is controlled from the state $x(0) = \begin{bmatrix} 0.8 \\ -0.15 \end{bmatrix} \in \tilde{P}_x^2$ to the origin of the state space, while fulfilling the state and input constraints.
6. Conclusions

We have presented a methodology for designing reference tracking controllers for constrained, discrete-time piecewise affine systems. The approach follows the idea of reference governor techniques where the desired set-point is filtered by a system called the “reference governor”. We have shown that state and input constraints can be satisfied by properly modifying the reference in order to force the system to evolve through invariant sets and stabilizable sets of the state and reference space.

The proposed technique is effective where nonlinear or mixed-integer online optimization is prohibitive for the given process computation infrastructure. Compared to the infinite time optimal solution, the approach presented in this paper is less computationally demanding at the price of suboptimality and smaller region of attraction.
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References


