Chapter 6

LIE GROUPS

6.1 Definitions and examples

Recall that a set G together with an operation $*$ (usually called multiplication) is called a *group* if the following four properties gold true:

(a) $x * y \in G, \forall x, y \in G$ (closure)

(b) $(x * y) * z = x * (y * z), \forall x, y, z \in G$ (associativity)

(c) there exists an identity element $e \in G$, such that $e * x = x * e = x, \forall x \in G$

(d) for any $x \in G$, there exists an inverse element x^{-1}, suich that $x * x^{-1} = x^{-1} * x = e$

If, in addition,

(e) $x * y = y * x$ (commutativity)

then G with the operation $*$ is called an *Abelian group*.

Lie groups result from the marriage of the algebraic concept of a group and the differential-geometric notion of a smooth manifold.

Definition: A *Lie group* G is a smooth manifold which is also a group in such a way that both the multiplication

$$m : G \times G \to G; \qquad m(x, y) = x * y, \forall x, y \in G \tag{6.1}$$

and the inversion

$$i : G \to G; \qquad i(x) = x^{-1}, \forall x \in G \tag{6.2}$$

are of class C^∞. Recall here the notion of a C^∞ map between manifolds $\mathcal{M}$ and $\mathcal{N}$

$$\phi_{UV} = y_V \circ \phi \circ x_U^{-1} : \mathbb{R}^m \to \mathbb{R}^n \tag{6.3}$$

is of class C^∞.

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Lie groups are particularly useful in the study of symmetry.

Examples

(i) Take $G = GL(n, R) = \{\text{all } n \times n \text{ real matrices } A \text{ with } \det A \neq 0\}$ This is a group under standard matrix multiplication with $e = I$, the identity matrix, and A^{-1} the usual matrix inverse. However, $GL(n, \mathbb{R})$ is a subset of the space $M(n, \mathbb{R})$ of *all* $n \times n$ real matrices (which is obviously isomorphic to $\mathbb{R}^{n^2}$, a trivial manifold). Moreover, $GL(n, \mathbb{R})$ is an *open* subset of $M(n, \mathbb{R})$, which can be argued as follows:

The set $\mathbb{R} - \{0\}$ is open in the standard topology of the real line. Now, $\det A : M(n, \mathbb{R}) \to \mathbb{R}$ is a smooth (n^{th}-degree polynomial) map on $M(n, \mathbb{R})$ therefore, since the inverse image of an open set (like $\mathbb{R} - \{0\}$) under a continuous map (like $\det A$) is open, it follows that $GL(n, \mathbb{R})$ is an open subset of $M(n, \mathbb{R})$, hence an n^2-manifold, which is a submanifold of $M(n, \mathbb{R})$. Matrix multiplication is C^∞, since the entries of AB are polynomial in the components of A and B. C^∞ smoothness of $\boldsymbol{A}^{-1}$ follows by a similar (rational) polynomial argument recalling the rule of finding $\boldsymbol{A}^{-1}$.

(ii) Take $G = O(n, \mathbb{R}) = \left\{\text{all } n \times n \text{ real matrices } A, \text{ such that } A^{\mathrm{T}}A = AA^{\mathrm{T}} = I\right\}$. This is the *orthogonal group* of $n \times n$ matrices under the usual matrix multiplication. The four group properties ar easy to verify. To argue that $O(n, \mathbb{R})$ is a manifold, note that the condition

$$A^{\mathrm{T}}A = AA^{\mathrm{T}} = I \tag{6.4}$$

yields $\frac{1}{2}n(n+1)$ independent equations everywhere on $M(n, \mathbb{R})$. This implies that $O(n, \mathbb{R})$ is a $n^2 - \frac{n(n+1)}{2} = \frac{(n-1)n}{2}$-dimensional immersion to $M(n, \mathbb{R})$. It turns out that $O(n, \mathbb{R})$ consists of two "disconnected" subgroups (i.e., subsets with Lie group properties),

$$\begin{aligned} SO(n, \mathbb{R}) &= \{A \in O(n, \mathbb{R})\,; \det A = +1\} \qquad \text{(proper orthogonal)} \\ O^-(n, \mathbb{R}) &= \{A \in O(n, \mathbb{R})\,; \det A = -1\} \qquad \text{(improper orthogonal)} \end{aligned} \tag{6.5}$$

Connected manifold: cannot be written as the union of two disjoint open sets.

Above:

$$O\left(n,\mathbb{R}\right)=SO\left(n,\mathbb{R}\right)\cup O^{-}\left(n,\mathbb{R}\right);\qquad SO\left(n,\mathbb{R}\right)\cap O^{-}\left(n,\mathbb{R}\right)=\emptyset \tag{6.6}$$

Unless explicitly stated, all Lie groups will be assumed here to be connected.

(iii) Take the special Lie group $SO\left(2\right)$, written as

$$SO\left(2\right)=\left\{\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix};\qquad 0\leqslant\theta<2\pi\right\} \tag{6.7}$$

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$$\begin{aligned} f_1 &: [0,2\pi)\rightarrow S' \\ f_2 &: [0,2\pi)\rightarrow SO\left(2\right) \\ f &= f_2\circ f_1^{-1}\end{aligned} \tag{6.8}$$

Comparing the above to $S^1=\left\{\left(\cos\theta,\sin\theta\right);\qquad 0\leq\theta<2\pi\right\}$, it is easily concluded that $SO\left(2\right)$ and S^1 are equivalent, in the sense that they are two 1-manifolds related by a 1-1 onto function which is differentiable with differentiable inverse.
$SO\left(2\right)$: Abelian
$SO\left(n\right)$: not Abelian for $n>2$

(iv) Take the Lie group (special linear) $SL\left(n,\mathbb{R}\right)$ defined as

$$SL\left(n,\mathbb{R}\right)=\left\{A\in GL\left(n,\mathbb{R}\right),\qquad \det A=1\right\} \tag{6.9}$$

This is obviously an n^2-1-dimensional immersion to $M\left(n,\mathbb{R}\right)$.

(v) Take the set $E\left(n,\mathbb{R}\right)$ of all isometries in $\mathbb{R}^n$, namely

$$E\left(n,\mathbb{R}\right)=\left\{\left(\boldsymbol{A},\boldsymbol{c}\right);\qquad \boldsymbol{A}\in O\left(n,\mathbb{R}\right),\qquad \boldsymbol{c}\in\mathbb{R}^n\right\} \tag{6.10}$$

Then, given any point $\boldsymbol{x}\in\mathbb{R}^n$,

$$\boldsymbol{x}^1=\left(\boldsymbol{A},\boldsymbol{c}\right)\left(\boldsymbol{x}\right)=\boldsymbol{A}\boldsymbol{x}+\boldsymbol{c}=\boldsymbol{\tau}_{\boldsymbol{c}}\left(\boldsymbol{A}\boldsymbol{x}\right)=\boldsymbol{\tau}\boldsymbol{c}\boldsymbol{A}\left(\boldsymbol{x}\right)\neq\boldsymbol{A}\boldsymbol{\tau}_{\boldsymbol{c}}\left(\boldsymbol{x}\right) \tag{6.11}$$

effects an orthogonal transformation $\boldsymbol{A}$ followed by a translation $\boldsymbol{c}$. Why is this isometry?

$$\begin{aligned}\boldsymbol{g}\left(\boldsymbol{x}^1-\boldsymbol{y}^1,\boldsymbol{x}^1,\boldsymbol{y}^1\right) &= \left\{\boldsymbol{A}\left(\boldsymbol{x}-\boldsymbol{y}\right)\right\}\cdot\left\{\boldsymbol{A}\left(\boldsymbol{x}-\boldsymbol{y}\right)\right\} \\ &= \left(\boldsymbol{x}-\boldsymbol{y}\right)\cdot\left\{\boldsymbol{A}^*\boldsymbol{A}\left(\boldsymbol{x}-\boldsymbol{y}\right)\right\} \\ &= \left(\boldsymbol{x}-\boldsymbol{y}\right)\cdot\left(\boldsymbol{x}-\boldsymbol{y}\right) \\ &= \boldsymbol{g}\left(\boldsymbol{x}-\boldsymbol{y},\boldsymbol{x}-\boldsymbol{y}\right)\end{aligned} \tag{6.12}$$

since $\boldsymbol{A}^* = \boldsymbol{A}^{\mathrm{T}}$ in this case (Euclidean metric). Now, it is clear that $E\left(n,\mathbb{R}\right)$ is equivalent to $O\left(n,\mathbb{R}\right)\times\mathbb{R}^n$, which is clearly a manifold. To argue the Lie group properties note that

$$\begin{aligned}\left(\boldsymbol{\tau_c A}\right)\left(\boldsymbol{\tau_d B}\right)\left(\boldsymbol{x}\right) &= \left(\boldsymbol{\tau_c A}\right)\left(\boldsymbol{Bx}+\boldsymbol{d}\right) \\ &= \boldsymbol{A}\left(\boldsymbol{Bx}+\boldsymbol{d}\right)+\boldsymbol{c} \\ &= \boldsymbol{ABx}+\boldsymbol{Ad}+\boldsymbol{c} \\ &= \boldsymbol{\tau_{Ad+c}AB}\left(\boldsymbol{x}\right)\end{aligned} \tag{6.13}$$

so that

$$\left(\boldsymbol{\tau_c A}\right)\left(\boldsymbol{\tau_d B}\right) = \boldsymbol{\tau_{Ad+c}}\left(\boldsymbol{AB}\right) \tag{6.14}$$

which is obviously a C^∞ operation. Likewise noting that

$$\begin{aligned}\boldsymbol{\tau_c A}:\boldsymbol{x}\to\boldsymbol{Ax}+\boldsymbol{c} &= \left(\boldsymbol{\tau_c A}\right)\left(\boldsymbol{x}\right) \\ \left(\boldsymbol{\tau_c A}\right)^{-1}:\left(\boldsymbol{Ax}+\boldsymbol{c}\right)\to\boldsymbol{x} &= \boldsymbol{A}^{-1}\left[\left(\boldsymbol{Ax}+\boldsymbol{c}\right)-\boldsymbol{c}\right]\end{aligned} \tag{6.15}$$

write

$$\begin{aligned}\left(\boldsymbol{\tau_c A}\right)^{-1}\left(\boldsymbol{x}\right) &= \boldsymbol{A}^{-1}\left(\boldsymbol{x}-\boldsymbol{c}\right) \\ &= \boldsymbol{A}^{-1}\boldsymbol{x}-\boldsymbol{A}^{-1}\boldsymbol{c} \\ &= \boldsymbol{\tau}_{\left(-\boldsymbol{A}^{-1}\boldsymbol{c}\right)}\boldsymbol{A}^{-1}\left(vx\right) \\ &= \boldsymbol{\tau}_{\left(-\boldsymbol{A}^{\mathrm{T}}\boldsymbol{c}\right)}\boldsymbol{A}^{\mathrm{T}}\left(\boldsymbol{x}\right)\end{aligned} \tag{6.16}$$

hence

$$\left(\boldsymbol{\tau_c A}\right)^{-1} = \boldsymbol{\tau}_{\left(-\boldsymbol{A}^{\mathrm{T}}\boldsymbol{c}\right)}\boldsymbol{A}^{\mathrm{T}} \tag{6.17}$$

which, again, is obviously of class C^∞.

6.2 Invariant vector and covector fields

Let G be a Lie group and $x \in G$. Define the *left translation* ℓ_x of G by x as

$$\ell_x : G \to G; \qquad \ell_x(y) = x * y \tag{6.18}$$

and the *right translation* r_x of G by x as

$$r_x : G \to G; \qquad r_x(y) = y * x \tag{6.19}$$

The above mappings are linear and diffeomorphic with inverses $(\ell_x)^{-1} = \ell_{x^{-1}}$ and $(r_x)^{-1} = r_{x^{-1}}$, respectively.

Terminology: left/right translation often referred to as left/right multiplication (more descriptive).

Property: $\ell_x \circ r_y = r_y \circ \ell_x$ (obvious)
$r_x \circ r_y \neq r_y \circ r_x$ and $\ell_x \circ \ell_y \neq \ell_y \circ \ell_x$ (in general)

For any given Lie group G, there exist certain vector fields on it that are characterized by their invariance under group multiplication (in a sense that will be defined below)

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A vector field $\boldsymbol{X}$ on G is called *left-invariant* if for any $g \in G$

$$(\ell_g)_* \boldsymbol{X} = \boldsymbol{X} \tag{6.20}$$

namely if

$$(T_p \ell_g)\, \boldsymbol{X} \circ \ell_g^{-1} (g * p) = \boldsymbol{X}(g * p) \tag{6.21}$$

for all $p \in G$. This means that the vector field remains unchanged under *any* left translation. Likewise, a vector field $\boldsymbol{X}$ on G is called *right-invariant* if for any $g \in G$

$$(g_g) * \boldsymbol{X} = \boldsymbol{X} \tag{6.22}$$

namely

$$(T_p r_g)\, \boldsymbol{X} \circ r_g^{-1} (p * g) = \boldsymbol{X}(p * g) \tag{6.23}$$

A special feature of great significance in Lie groups is that for any vector $\boldsymbol{v} = \boldsymbol{X}(e)$ at e (the identity element of G),

$$(T_e \ell_g)\, \boldsymbol{X}(e) = \boldsymbol{X}(g * e) = \boldsymbol{X}(g) \tag{6.24}$$

namely that given any $\bar{\boldsymbol{X}} = \boldsymbol{X}(e)$, there is a unique left invariant vector field $\boldsymbol{X}$ on G with the value $\boldsymbol{X}(e) = \bar{\boldsymbol{X}}$ at e. Conversely, any left-invariant vector field can be fully determined by its value at e. Corresponding results apply to right-invariance.

Theorem: Every left invariant vector field $\boldsymbol{X}$ on a Lie group is of class C^∞.

Proof: Omitted.

Recalling that for time-independent vector field $\boldsymbol{Y}$

$$L_{\boldsymbol{X}}\boldsymbol{Y} = [\boldsymbol{X}, \boldsymbol{Y}] \tag{6.25}$$

where $[\boldsymbol{X}, \boldsymbol{Y}]$ is the Lie bracket of $\boldsymbol{X}$ and $\boldsymbol{Y}$. Also, recalling from homework that

$$\phi_* L_{\boldsymbol{X}}\boldsymbol{Y} = L_{\phi_*\boldsymbol{X}}(\phi_*\boldsymbol{Y}) \tag{6.26}$$

it follows that, if $\boldsymbol{X}$, $\boldsymbol{Y}$ are left-invariant, then

$$\begin{aligned}(\ell_g)_* L_{\boldsymbol{X}}\boldsymbol{Y} &= (\ell_g)_* [\boldsymbol{X}, \boldsymbol{Y}] \\ &= L_{(\ell_g)_*\boldsymbol{X}}\big((\ell_g)_*\boldsymbol{Y}\big) \\ &= \big[(\ell_g)_*\boldsymbol{X}, (\ell_g)_*\boldsymbol{Y}\big] \\ &= [\boldsymbol{X}, \boldsymbol{Y}]\end{aligned} \tag{6.27}$$

Therefore $[\boldsymbol{X}, \boldsymbol{Y}]$ is left-invariant if $\boldsymbol{X}$, $\boldsymbol{Y}$ are left-invariant!

The notions of left- and right-invariance can be readily extended from vector fields to covector fields.

A covector field $\boldsymbol{v}^*$ on G is called *left-invariant* if, for any $g \in G$,

$$(\ell_g)^* \boldsymbol{v}^* = \boldsymbol{v}^*, \qquad \{\text{recall } \langle \phi^*\boldsymbol{v}^*, \boldsymbol{z}\rangle = \langle \boldsymbol{v}^*, \phi_*\boldsymbol{z}\rangle \tag{6.28}$$

namely if

$$(T_p\ell_g)^* \boldsymbol{v}^* \circ \ell_g(p) = \boldsymbol{v}^*(p) \tag{6.29}$$

for all $p \in G$.

Likewise, a covector field $\boldsymbol{v}^*$ on G is called *right-invariant* if, for any $g \in G$

$$(r_g)^* \boldsymbol{v}^* = \boldsymbol{v}^* \tag{6.30}$$

namely if

$$(T_p r_g)^* \boldsymbol{v}^* \circ r_g(p) = \boldsymbol{v}^*(p) \tag{6.31}$$

for all $p \in G$.

Again, noting that

$$(T_e\ell_g)^* \boldsymbol{v}^*(g) = \boldsymbol{v}^*(e) \tag{6.32}$$

one may build a left-invariant covector field $\boldsymbol{v}^*$ from the above property by only knowing its value at e. This field is unique, for a given $\boldsymbol{v}^*(e)$. Conversely, any left-invariant covector field is fully determined by its value at e. Similar definitions of invariance apply to $\binom{s}{r}$-type tensors.

6.3 Lie algebra of a Lie group

Generally speaking, a vector space $\{V, +; \mathbb{R}, \cdot\}$ is a *Lie algebra* if, in addition to its vector space structure (9 properties), it possesses a product map $[\cdot,\cdot] : V \times V \to V$ with the following properties:

(a) $[\alpha\boldsymbol{u}_1 + \beta\boldsymbol{u}_2, \boldsymbol{v}] = \alpha[\boldsymbol{u}_1, \boldsymbol{v}] + \beta[\boldsymbol{u}_2, \boldsymbol{v}]$
$[\boldsymbol{u}, \alpha\boldsymbol{v}_1 + \beta\boldsymbol{v}_2] = \alpha[\boldsymbol{u}, \boldsymbol{v}_1] + \beta[\boldsymbol{u}, \boldsymbol{v}_2]$
for all $\alpha, \beta \in \mathbb{R}$, $\boldsymbol{u}, \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{v}, \boldsymbol{v}_1, \boldsymbol{v}_2 \in V$ (bilinearity)

(b) $[\boldsymbol{u}, \boldsymbol{v}] = -[\boldsymbol{v}, \boldsymbol{u}]$ for all $\boldsymbol{u}, \boldsymbol{v} \in V$ (anti-commutativity)

(c) $[\boldsymbol{u}, [\boldsymbol{v}, \boldsymbol{w}]] + [\boldsymbol{v}, [\boldsymbol{w}, \boldsymbol{u}]] + [\boldsymbol{w}, [\boldsymbol{u}, \boldsymbol{v}]] = \boldsymbol{0}$ for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V$ (Jacobi identity property)

Returning to the case of a Lie group G, consider the vector space of *all* left-invariant vector fields on G. This space is (almost) universally denoted by $\mathfrak{g}$. Note that all 9 properties of a vector space can be easily checked for the case of $\mathfrak{g}$, e.g., if $\boldsymbol{X}$ and $\boldsymbol{Y}$ are left-invariant vector fields on G, then

$$(\ell_g)_*[\alpha\boldsymbol{X} + \beta\boldsymbol{Y}] = \alpha(\ell_g)_*\boldsymbol{X} + \beta(\ell_g)_*\boldsymbol{Y} = \alpha\boldsymbol{X} + \beta\boldsymbol{Y} \tag{6.33}$$

hence $\alpha\boldsymbol{X} + \beta\boldsymbol{Y}$ is also left-invariant (closure property). It is also clear that since any left-invariant vector field $\boldsymbol{X}$ is completely determined by $\boldsymbol{X}(e)$, then we can identify $\mathfrak{g}$ with the tangent space to G at the identity e, i.e.,

$$\mathfrak{g} \equiv T_eG \tag{6.34}$$

Furthermore, $\mathfrak{g}$ becomes a Lie algebra when equipped with the usual Lie bracket $[\cdot,\cdot]$, since the Lie bracket of two left-invariant vector field is left-invariant, and it satisfies all three properties in the Lie algebra definition.

Remark: Some authors take the Lie algebra to consist of all right- (as opposed to left-) invariant vector fields on G. This is largely a matter of taste.

Example: Deduce the Lie algebra $\mathfrak{gl}\left(n,\mathbb{R}\right)$ of the Lie group $GL\left(n,\mathbb{R}\right)$. Recall that $GL\left(b,\mathbb{R}\right)$ is an open subset of $M\left(n,\mathbb{R}\right)\equiv\mathbb{R}^{n^2}$, hence it is a n^2-manifold. Since,

$$\mathfrak{gl}\left(n,\mathbb{R}\right)\equiv T_e GL\left(n,\mathbb{R}\right)=T_{\boldsymbol{x}} GL\left(n,\mathbb{R}\right) \tag{6.35}$$

it follows that $\mathfrak{gl}\left(n,\mathbb{R}\right)=\mathbb{R}^{n^2}\equiv M\left(n,\mathbb{R}\right)$! (Think of $T_p\mathcal{M}$ where $\mathcal{M}$ is open in $\mathbb{R}^n$).

To determine the particular expression for the Lie bracket, recall that

$$\left[\boldsymbol{X},\boldsymbol{Y}\right]^i=\frac{\partial Y^i}{\partial x^j}X^j-\frac{\partial X^i}{\partial x^j}Y^j \tag{6.36}$$

where $\boldsymbol{X},\boldsymbol{Y}\in T_{\boldsymbol{x}}GL\left(n,\mathbb{R}\right)$. Now, write

$$\left[\boldsymbol{X}_{\boldsymbol{A}}\left(\boldsymbol{I}\right),\boldsymbol{X}_{\boldsymbol{B}}\left(\boldsymbol{I}\right)\right]=\left[\boldsymbol{A},\boldsymbol{B}\right] \tag{6.37}$$

where $\boldsymbol{X}_{\boldsymbol{A}}\left(\boldsymbol{I}\right),\boldsymbol{X}_{\boldsymbol{B}}\left(\boldsymbol{I}\right)\in T_{\boldsymbol{I}}GL\left(n,\mathbb{R}\right)$ and $\boldsymbol{A},\boldsymbol{B}\in\mathfrak{gl}\left(n,\mathbb{R}\right)$ and the vector fields $\boldsymbol{X}_A$ and $\boldsymbol{X}_B$ are defined simply as

$$\boldsymbol{X}_A\left(\boldsymbol{Y}\right)=\boldsymbol{Y}\boldsymbol{A},\qquad \boldsymbol{X}_B\left(\boldsymbol{Y}\right)=\boldsymbol{Y}\boldsymbol{B} \tag{6.38}$$

and are left-invariant vector fields on $GL\left(n,\mathbb{R}\right)$, because

$$\begin{aligned}
\boldsymbol{X}_A\left(\ell_{\boldsymbol{Z}}\boldsymbol{Y}\right)&=\boldsymbol{X}_A\left(\boldsymbol{Z}\boldsymbol{Y}\right)\\
&=\boldsymbol{Z}\boldsymbol{Y}\boldsymbol{A}\\
&=\ell_{\boldsymbol{Z}}\boldsymbol{X}_A\left(\boldsymbol{Y}\right)\\
&=T_{\boldsymbol{Y}}\ell_{\boldsymbol{Z}}\boldsymbol{X}_A\left(\boldsymbol{Y}\right)\\
&=\left(\ell_{\boldsymbol{Z}}\right)_*\boldsymbol{X}_A\left(\boldsymbol{Y}\right)
\end{aligned} \tag{6.39}$$

for all $\boldsymbol{Y},\boldsymbol{Z}\in GL\left(n,\mathbb{R}\right)$, given that $\ell_{\boldsymbol{Z}}$ is a linear function in its argument. Returning to the Lie bracket, note that since $\boldsymbol{X}_A\left(\boldsymbol{Y}\right)$ is also linear in its argument

$$\begin{aligned}
\left[\boldsymbol{A},\boldsymbol{B}\right]&=\left[\boldsymbol{X}_A\left(\boldsymbol{I}\right),\boldsymbol{X}_B\left(\boldsymbol{I}\right)\right]\\
&=\left.\frac{\partial\boldsymbol{X}_B\left(\boldsymbol{Y}\right)}{\partial\boldsymbol{Y}}\right|_{\boldsymbol{Y}=\boldsymbol{I}}\boldsymbol{X}_A\left(\boldsymbol{I}\right)-\left.\frac{\partial\boldsymbol{X}_A\left(\boldsymbol{Y}\right)}{\partial\boldsymbol{Y}}\right|_{\boldsymbol{Y}=\boldsymbol{I}}\boldsymbol{X}_B\left(I\right)\\
&=\boldsymbol{X}_B\left(\boldsymbol{X}_A\left(\boldsymbol{I}\right)\right)-\boldsymbol{X}_A\left(\boldsymbol{X}_B\left(\boldsymbol{I}\right)\right)\\
&=\boldsymbol{X}_B\left(\boldsymbol{A}\right)=\boldsymbol{X}_A\left(\boldsymbol{B}\right)\\
&=\boldsymbol{A}\boldsymbol{B}-\boldsymbol{B}\boldsymbol{A}\qquad\text{(the \textit{matrix commutator})}
\end{aligned} \tag{6.40}$$

Therefore, the space of left-invariant vector fields on the Lie group $GL\left(n,\mathbb{R}\right)$ is rendered a Lie algebra by means of the matrix commutator Lie bracket. If we had defined the Lie algebra using right invariance, then we would have found $\left[\boldsymbol{A},\boldsymbol{B}\right]=\boldsymbol{B}\boldsymbol{A}-\boldsymbol{A}\boldsymbol{B}$.

6.4 One-parameter subgroups and the exponential map

Given two Lie groups G and H, the map $\phi : G \to H$ is termed a *homeomorphism* if

$$\phi\left(g_1 *' g_2\right) = \phi\left(g_1\right) * \phi\left(g_2\right) \tag{6.41}$$

for all $g_1, g_2 \in G$. Here, $*'$ and $*$ are the group multiplication operators in G and H, respectively. Clearly, the product operations on G and H are invariant under a homomorphic map.

Remark: homomorphism $\neq$ homeomorphism

A homomorphism $\phi : \mathbb{R} \to G$ is called a *one-parameter subgroup of* G. This is a subset of G which is closed under group multiplication. Here, $\mathbb{R}$ is an "additive" group, i.e.,

$$s * t = s + t = t + s \tag{6.42}$$

so that

$$\begin{aligned} \phi\left(s+t\right) &= \psi\left(s\right) * \phi\left(t\right) = \phi\left(s\right)\phi\left(t\right) \qquad \text{(ignore $*$ for brevity)} \\ &= \phi\left(t+s\right) = \phi\left(t\right)\phi\left(s\right) \end{aligned} \tag{6.43}$$

Example 1: Take $G = \mathbb{R}$ be again the "additive" group and assume that ϕ is of class C^1. Then

$$\phi\left(s+t\right) = \phi\left(s\right) + \phi\left(t\right) \tag{6.44}$$

hence

$$\frac{d\phi\left(s+t\right)}{ds} = \frac{d\phi\left(s\right)}{ds} \tag{6.45}$$

so for taking the limit as $s \to 0$

$$\phi'\left(t\right) = \phi'\left(0\right) \tag{6.46}$$

for all $t \in \mathbb{R}$. This and the preceding homomorphic relation imply that

$$\phi\left(t\right) = ct; \qquad c = \phi'\left(0\right) \tag{6.47}$$

Example 2: Take $G = \mathbb{R} - \{0\}$ with the usual multiplication as the group operation. Then if ϕ is assumed of class C^∞, then

$$\phi\left(s+t\right) = \phi\left(s\right)\phi\left(t\right) \tag{6.48}$$

hence

$$\frac{d\phi\left(s+t\right)}{ds} = \frac{d\phi\left(s\right)}{ds}\phi\left(t\right) \tag{6.49}$$

and in the limit as $s \to 0$

$$\phi'(t) = \phi'(0)\,\phi(t) \tag{6.50}$$

Putting $t = 0$ above gives rise to

$$\phi(0) = 1 \tag{6.51}$$

Therefore, we are seeking a function $\phi(t)$ of class C^∞ satisfying (6.50), (6.51). The solution is

$$\phi(t) = e^{ct}; \qquad \phi'(0) \tag{6.52}$$

Note that $\mathbb{R} - \{0\} = GL(1, \mathbb{R})$. Extending the above analysis to $G = GL(n, \mathbb{R})$ gives rise to

$$\begin{aligned} \boldsymbol{\phi}'(t) &= \boldsymbol{\phi}'(0)\,\boldsymbol{\phi}(t) \\ \boldsymbol{\phi}(0) &= \boldsymbol{I} \end{aligned} \tag{6.53}$$

where the above are equations involving $n \times n$ matrices. The solution to these equations is

$$\boldsymbol{\phi}(t) = e^{\boldsymbol{\phi}'(0)t} \tag{6.54}$$

where the matrix exponential is defined as

$$e^{\boldsymbol{A}} = \boldsymbol{I} + \boldsymbol{A} + \frac{\boldsymbol{A}^2}{2!} + \ldots + \frac{\boldsymbol{A}^n}{n!} + \ldots \tag{6.55}$$

This series can be shown to be convergent.

Indeed, given $\phi(t)$ as before, it follows that

$$\begin{aligned} \boldsymbol{\phi}'(t) &= \lim_{s \to 0} \frac{e^{(t+s)\phi'(0)} - e^{t\phi'(0)}}{s} \\ &= \lim_{s \to 0} \frac{\left(e^{s\boldsymbol{\phi}'(0)} - \boldsymbol{I}\right)}{s} e^{t\boldsymbol{\phi}'(0)} \\ &= \lim_{s \to 0} \frac{s\boldsymbol{\phi}'(0) + \frac{s^2}{2!}\phi''(0) + \ldots}{s} e^{t\phi'(0)} \\ &= \boldsymbol{\phi}'(0)\,\boldsymbol{\phi}(t) \end{aligned} \tag{6.56}$$

so also

$$\boldsymbol{\phi}(0) = \boldsymbol{I} \tag{6.57}$$

The previous results for $GL(n, \mathbb{R})$ have a very simple generalization to any group G. Take a C^∞ homomorphism $\phi : \mathbb{R} \to G$, so that

$$\phi(t + s) = \phi(t)_*\,\phi(s) \tag{6.58}$$

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from where

$$\begin{aligned}\phi'(t) &= \phi(t)_* \phi'(0) \\ &= \ell_{\phi(t)}\phi'(0) \\ &= \left(\ell_{\phi(t)}\right)_* \phi'(0)\end{aligned} \tag{6.59}$$

since $\boldsymbol{\ell}_x$ is a linear function. The equation

$$\phi'(t) = \left(\ell_{\phi(t)}\right)_* \phi'(0) \tag{6.60}$$

reveals that the tangent vector field ϕ' to the one-parameter subgroup ϕ is left-invariant along the subgroup. Hence, given a vector $\phi'(0) \in T_eG$, the one-parameter subgroup of G whose tangent at e is $\phi'(0)$ is the integral curve through e of the vector field on G resulting from left translation of $\phi'(0)$ on G.

Now, for *any* Lie group G, we may define the *exponential map* $\exp : g \to G$ as follows: starting with an $\boldsymbol{v} \in g$, let $\phi_{\boldsymbol{v}}$ be the (unique) C^∞ homomorphism such that

$$\phi'_{\boldsymbol{v}}(0) = \boldsymbol{v} \tag{6.61}$$

Then,

$$\exp(t\boldsymbol{v}) = \phi_{\boldsymbol{v}}(t) \tag{6.62}$$

and

$$\exp(\boldsymbol{v}) = \phi_{\boldsymbol{v}}(1) \tag{6.63}$$

Example Exponential map in $SO(3,\mathbb{R})$.

Recall that

$$SO(3,\mathbb{R}) = \left\{\boldsymbol{Q} \in GL(3,\mathbb{R}), \qquad \boldsymbol{Q}^{\mathrm{T}}\boldsymbol{Q} = \boldsymbol{Q}\boldsymbol{Q}^{\mathrm{T}} = \boldsymbol{I}, \qquad \det \boldsymbol{Q} = +1\right\} \tag{6.64}$$

is a Lie group under the usual matrix multiplication (it is a 3-dimensional immersion to $M(3,\mathbb{R}) \equiv \mathbb{R}^9$). By definition, the set of all left-invariant vector fields in $SO(3,\mathbb{R})$ is

$$\mathfrak{so}(3,\mathbb{R}) = T_{\boldsymbol{I}}(SO(3,\mathbb{R})) \tag{6.65}$$

To understand what is the content of $\mathcal{so}(3,\mathbb{R})$, take a curve $\hat{\boldsymbol{Q}} : \mathbb{R} \to SO(3,\mathbb{R})$

$$\boldsymbol{Q} = \hat{\boldsymbol{Q}}(t) \tag{6.66}$$

such that $\boldsymbol{Q}(0) = \boldsymbol{I}$ and find the rate of change of $\boldsymbol{Q}$ by noting that

$$\boldsymbol{Q}^{\mathrm{T}}(t)\,\boldsymbol{Q}(t) = \boldsymbol{I} \text{ for all } t \in \mathbb{R} \tag{6.67}$$

hence

$$\dot{\boldsymbol{Q}}^{\mathrm{T}}(t)\,\boldsymbol{Q}(t) + \boldsymbol{Q}^{\mathrm{T}}(t)\,\dot{\boldsymbol{Q}}(t) = \boldsymbol{0} \tag{6.68}$$

Evaluating this at the identity lead to

$$\dot{\boldsymbol{Q}}^{\mathrm{T}}(\boldsymbol{I}) + \dot{\boldsymbol{Q}}(\boldsymbol{I}) = \boldsymbol{0} \tag{6.69}$$

therefore

$$\mathcal{so}(3,\mathbb{R}) = \left\{\boldsymbol{\Omega} \in M(3,\mathbb{R}), \qquad \boldsymbol{\Omega} + \boldsymbol{\Omega}^{\mathrm{T}} = \boldsymbol{0}\right\} \tag{6.70}$$

ie.e, it consists of all skew-symmetric 3 × 3 matrices. Note here that $\mathcal{so}(3,\mathbb{R})$ is isomorphic to $\mathbb{R}^3$. Clearly, $\mathcal{so}(3,\mathbb{R})$ is a Lie algebra under the standard matrix commutator

$$[\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2] = \boldsymbol{\Omega}_1\boldsymbol{\Omega}_2 - \boldsymbol{\Omega}_2\boldsymbol{\Omega}_1 = \boldsymbol{\Omega}_1\boldsymbol{\Omega}_2 - (\boldsymbol{\Omega}_1\boldsymbol{\Omega}_2)^{\mathrm{T}} \tag{6.71}$$

The exponential map $\exp : \mathcal{so}(3,\mathbb{R}) \to SO(3,\mathbb{R})$ is defined according to the previous discussion by taking any $\Omega \in \mathcal{so}(3,\mathbb{R})$ and letting $\boldsymbol{\phi} : \mathbb{R} \to SO(3,\mathbb{R})$ be a C^∞ homomorphism such that $\dot{\boldsymbol{\phi}}(0) = \boldsymbol{\Omega}$. Clearly,

$$\begin{aligned}\dot{\boldsymbol{\phi}}(t) &= \dot{\boldsymbol{\phi}}(0)\,\boldsymbol{\phi}(t)\\ &= \boldsymbol{\Omega}\boldsymbol{\phi}(t)\end{aligned} \tag{6.72}$$

and $\phi(0) = \boldsymbol{I}$ for $\boldsymbol{\phi}$ to be a homomorphism. Now

$$\begin{aligned}\exp(\boldsymbol{\Omega}) &= \phi(1)\\ &= \boldsymbol{I} + \boldsymbol{\Omega} + \frac{1}{2!}\boldsymbol{\Omega}^2 + \dots\end{aligned} \tag{6.73}$$

and $\exp(t\boldsymbol{\Omega}) = \boldsymbol{\phi}(t)$. Amazingly, there exist closed from expressions for the exponential map of skew-symmetric matrices in $M(3,\mathbb{R})$. One of them is

$$\exp(\boldsymbol{\Omega}) = \boldsymbol{I} + \frac{\sin\|\boldsymbol{\omega}\|}{\|\boldsymbol{\omega}\|}\boldsymbol{\Omega} + \frac{1}{2}\left[\frac{\sin\left(\frac{\|\boldsymbol{\omega}\|}{2}\right)}{\frac{\|\boldsymbol{\omega}\|}{2}}\right]^2\boldsymbol{\Omega}^2 \tag{6.74}$$

where $\boldsymbol{\omega}$ is the axial vector of $\boldsymbol{\Omega}$, i.e. for any $\boldsymbol{z} \in \mathbb{R}^3$

$$\boldsymbol{\Omega z} = \boldsymbol{\omega} \times \boldsymbol{z} \tag{6.75}$$

This is a reformulation of the classical Podrigues formula. Hence, there is no need to approximate the infinite series which defines the exponential of a matrix. The exponential map in $SO(3, \mathbb{R})$ has application in the numerical solution of evolution equations of the form

$$\begin{aligned} \dot{\boldsymbol{Q}}(t) &= \boldsymbol{\Omega}(t)\,\boldsymbol{Q}(t) \\ \boldsymbol{Q}(0) &= \boldsymbol{I} \end{aligned} \tag{6.76}$$

where $\boldsymbol{\Omega}$ is a given skew-symmetric matrix function of time. In the time interval $(t_n, t_{n+1}]$ we can reformulate the above problem as

$$\begin{aligned} \dot{\boldsymbol{Q}}(t) &= \boldsymbol{\Omega}(t)\,\boldsymbol{Q}(t) \\ \boldsymbol{Q}(t_n) &= \boldsymbol{Q}_n \end{aligned} \tag{6.77}$$

or

insert figures

$$\begin{aligned} \dot{\boldsymbol{Q}}_1(t) &= \boldsymbol{\Omega}(t)\,\boldsymbol{Q}_1(t) \\ \boldsymbol{Q}_1(t_n) &= \boldsymbol{I} \end{aligned} \tag{6.78}$$

where $\boldsymbol{Q}_1(t) = \boldsymbol{Q}(t)\,\boldsymbol{Q}_n^{-1}$. It follows that the exponential update is

$$\boldsymbol{Q}_1(t) = \exp\left(\int_{t_n}^{t} \boldsymbol{\Omega}(t)\,dt\right) \tag{6.79}$$

or

$$\boldsymbol{Q}(t) = \exp\left(\int_{t_n}^{t} \boldsymbol{Q}(t)\,dt\right) \boldsymbol{Q}_n \tag{6.80}$$

Therefore, a generalized mid-point rule for the exponential update can be deduced in the form

$$\boldsymbol{Q}_{n+1} = \exp\left(\Delta t_n \boldsymbol{\Omega}(t_{n+\alpha})\right) \boldsymbol{Q}_n \tag{6.81}$$

where $\Delta t_n = t_{n+1} - t_n$ and $t_{n+\alpha} = (1-\alpha)\, t_n + \alpha t_{n+1}$.

Special cases:

$\alpha = 0$: forward Euler
$\alpha = \frac{1}{2}$: midpoint
$\alpha = 1$: backward Euler