ME280B Midterm Examination

March 31st, 11:10–12:30

NAME : ______________________________

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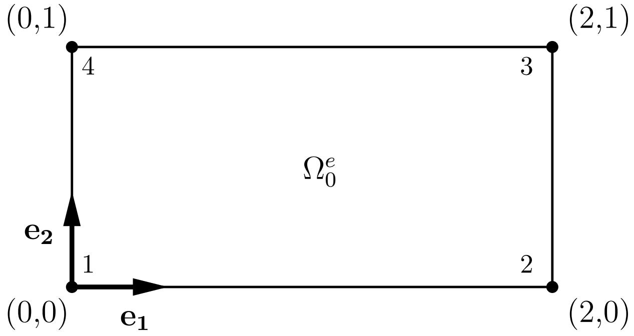
Problem 1: ____ /20 points

Problem 2: ____ /15 points

Problem 3: ____ /15 points

Problem 1 (3+4+3+3+3+4 points)

A 4-noded finite element is used in the analysis of a solid body in plane strain. In the reference configuration, the element occupies a rectangular region Ω_0^e, as shown in the figure below.



(a) Write the standard shape functions N_I, $I = 1$–4, for the element in terms of the (X_1, X_2)-coordinates corresponding to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2\}$.

(b) Assuming an isoparametric mapping, determine the components of the deformation gradient $\mathbf{F}$ with respect to the (common) basis $\{\mathbf{e}_1, \mathbf{e}_2\}$, as a function of the 8 displacement degrees-of-freedom (u_{I1}, u_{I2}), $I = 1$–4, when

$$\begin{aligned} u_{11} &= u_{12} = 0 \quad , \quad u_{21} = \bar{u} \; , \; u_{22} = 0 \; , \\ u_{31} &= \bar{u} \; , \; u_{32} = -\bar{v} \quad , \quad u_{41} = 0 \; , \; u_{42} = -\bar{v} \; . \end{aligned}$$

(c) Calculate the Jacobian J of the deformation gradient and place restrictions on $\bar{u}$ and $\bar{v}$ such that $J > 0$.

(d) Draw a sketch of the current configuration of the element.

(e) Suppose that the edge $2-3$ of the above element is subjected to an applied traction $\bar{\mathbf{p}}$ per unit length of the <u>reference</u> configuration, given by $\bar{\mathbf{p}} = p\mathbf{e}_1$, where p is a constant. Calculate the equivalent nodal forces on nodes 2 and 3.

(f) Suppose now that the edge $2-3$ is subjected to an applied traction $\bar{\mathbf{t}}$ per unit length of the <u>current</u> configuration, such that $\bar{\mathbf{t}} = p\mathbf{e}_1$, where p is a constant. Calculate again the equivalent nodal forces on nodes 2 and 3. Are the two sets of equivalent nodal forces calculated in part (e) and (f) equal to each other or not? Explain your answer.

Problem 2 (15 points)

The weak form of the balance of linear momentum can be written for a typical finite element with domain Ω^e and boundary $\partial\Omega^e$ as

$$\begin{aligned}\int_{\Omega^e} \boldsymbol{\xi} \cdot \rho \ddot{\mathbf{u}} \, dv \ + \ \int_{\Omega^e} \frac{\partial \boldsymbol{\xi}}{\partial \mathbf{x}} \cdot \mathbf{T} \, dv \\ = \int_{\Omega^e} \boldsymbol{\xi} \cdot \rho \mathbf{b} \, dv \ + \ \int_{\partial\Omega^e \cap \Gamma_q} \boldsymbol{\xi} \cdot \bar{\mathbf{t}} \, da \ + \ \int_{\partial\Omega^e - \Gamma_q \cap \partial\Omega^e} \boldsymbol{\xi} \cdot \bar{\mathbf{t}} \, da \ . \quad (\dagger)\end{aligned}$$

In the above equation, ρ is the mass density, $\mathbf{T}$ is the Cauchy stress tensor, $\mathbf{b}$ is the body force per unit mass, $\mathbf{t}$ is the traction vector, and $\bar{\mathbf{t}}$ is the prescribed traction on the part Γ_q of the exterior boundary of the body. Also, $\boldsymbol{\xi}$ is an arbitrary tangent vector on the current configuration of the body.

Assume that the prescribed traction vector is of the form

$$\bar{\mathbf{t}} \ = \ -p \, \mathbf{n} \ ,$$

where $\mathbf{n}$ is the outward unit normal to Γ_q and p is a constant. Show that the differential of the boundary traction integral term in (†) along the direction $\Delta\mathbf{u}$, with respect to the current configuration, can be written as

$$D\Big[\int_{\partial\Omega^e \cap \Gamma_q} \boldsymbol{\xi} \cdot \bar{\mathbf{t}} \, da\Big](\mathbf{u}, \Delta\mathbf{u}) \ = \ -p \int_{\partial\Omega^e \cap \Gamma_q} \boldsymbol{\xi} \cdot \Big[\{\operatorname{div}(\Delta\mathbf{u})\}\mathbf{n} \ - \ \Big(\frac{\partial \Delta\mathbf{u}}{\partial \mathbf{x}}\Big)^T \mathbf{n}\Big] \, da \ .$$

Problem 3 (6+3+6 points)

Consider a material whose constitutive response is of the form

$$\dot{\mathbf{T}} \;=\; \mathbf{W}\mathbf{T} - \mathbf{T}\mathbf{W} + \alpha\mathbf{D} + \beta\mathbf{D}^2 \;, \tag{‡}$$

where $\mathbf{T}$ is the Cauchy stress tensor, $\mathbf{D}$ is the rate-of-deformation tensor, $\mathbf{W}$ is the vorticity tensor, and α, β are constants.

(a) Show that the constitutive relation (‡) is objective.

(b) Assume that an arbitrary Lagrangian-Eulerian (ALE) finite element method is employed to model a deformable body that obeys the relation (‡). Argue (without using equations!) why this material law cannot be directly used in an ALE method when expressed as in (‡).

(c) Show that the constitutive relation (‡) can be also written as

$$\frac{d_{M}\mathbf{T}}{dt} \;=\; \frac{\partial \mathbf{T}}{\partial \mathbf{x}_{M}} \cdot (\mathbf{v}_{M} - \mathbf{v}) + \mathbf{W}\mathbf{T} - \mathbf{T}\mathbf{W} + \alpha\mathbf{D} + \beta\mathbf{D}^2 \;,$$

where $\mathbf{x}_{M}$ and $\mathbf{v}_{M}$ are the mesh position vector and mesh velocity vector, respectively.