# ME280A Midterm Examination

*October 14th, 17:10–18:30*

NAME : ______________________________

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Problem 1: _____ /25 points

Problem 2: _____ /35 points

Problem 3: _____ /40 points

**Problem 1** (5+20 points)

Consider the boundary-value problem

$$\begin{aligned} \frac{du}{dx} + u &= 0 \quad \text{in } \Omega = (-1,1)\ , \\ u(-1) &= 1\ , \end{aligned}$$

and assume a polynomial approximation $u_h$ of the dependent variable $u$ in the form

$$u_h(x) \;=\; \alpha_0\phi_0(x) \;+\; \alpha_1\phi_1(x) \;+\; \dots\ ,$$

where $\alpha_i$, $i = 0, 1, \dots$ are parameters to be determined. Here, the polynomials $\phi_i(x)$, $i = 0, 1, \dots$, are defined by the recursive relation

$$\phi_{i+2}(x) \;=\; 2x\phi_{i+1}(x) - \phi_i(x) \quad , \quad i = 0, 1, \dots\ ,$$

with $\phi_0(x) = 1$ and $\phi_1(x) = x$. These functions are called *Chebyshev* polynomials and satisfy the orthogonality property

$$\int_{-1}^{1} \frac{\phi_i\phi_j}{\sqrt{1-x^2}}\,dx \;=\; \begin{cases} \pi & \text{if} \quad i = j = 0 \\ \pi/2 & \text{if} \quad i = j \neq 0 \\ 0 & \text{if} \quad i \neq j \end{cases}\ .$$

(a) Start with a two-parameter Chebyshev polynomial approximation of $u_h$ and reduce it to a one-parameter approximation by directly enforcing the boundary condition.

(b) Starting from the reduced form of $u_h(x)$ in part (a), find an approximate solution to the boundary-value problem using a Petrov-Galerkin method. In order to take advantage of the orthogonality of Chebyshev polynomials, choose the weighting function $w_h$ to be

$$w_h(x) \;=\; \beta\,\frac{\phi_0(x)}{\sqrt{1-x^2}}\ ,$$

where $\beta$ is an arbitrary scalar.

**Problem 2** (5+10+20 points)

Consider the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0 \quad \text{for } (x,t) \in (0,1) \times (0,T) \ , \tag{1}$$

subject to boundary conditions

$$u(0,t) = 0 \quad \text{for } \ t \in (0,T) \ , \tag{2}$$

$$u(1,t) = 0 \quad \text{for } \ t \in (0,T) \ , \tag{3}$$

where $T > 0$ is a given constant, and initial conditions

$$u(x,0) = \begin{cases} x & \text{for } \ 0 \leq x \leq \frac{1}{2} \\ 1 - x & \text{for } \ \frac{1}{2} < x \leq 1 \end{cases} \tag{4}$$

and

$$\frac{\partial u}{\partial t}(x,0) = 0 \quad \text{for } x \in (0,1) \ . \tag{5}$$

Let a family of approximations $u_h(x,t)$ be written as

$$u_h(x,t) = \{\alpha_0 + \alpha_1 x + \alpha_2 x^2\} \, \varphi(t) \ , \tag{6}$$

where $\varphi(t)$ is a (yet unknown) function of time, and $\alpha_0$, $\alpha_1$ and $\alpha_2$ are scalar parameters to be determined.

(a) Obtain a reduced form of $u_h(x,t)$ by enforcing boundary conditions (2) and (3).

(b) Determine initial conditions $\varphi(0)$ and $\varphi'(0)$ for the function $\varphi(t)$ using a one-point domain collocation method on the reduced form of $u_h(x,t)$ to approximately satisfy equations (4) and (5). Choose the mid-point of the domain as the collocation point.

(c) Arrive at a second-order ordinary differential equation for $\varphi(t)$ by applying to differential equation (1) a Petrov-Galerkin method <u>in the spatial domain</u>. Use the approximation function $u_h(x,t)$ of part (a) and a weighting function $w_h(x,t)$ given by

$$w_h(x,t) = \beta x \ ,$$

where $\beta$ is an arbitrary scalar parameter. Find a closed-form expression for $\varphi(t)$ by solving the differential equation analytically and using the initial conditions obtained in part (b). Finally, substitute $\varphi(t)$ into equation (6) to obtain $u_h$.

**Problem 3** (15+15+5+5 points)

Consider the functional

$$I[u] \;=\; \int_0^1 \left\{ \left(\frac{d^2u}{dx^2}\right)^2 + \left(\frac{du}{dx}\right)^2 + u^2 \right\} dx \;.$$

(a) Determine the differential equation whose solution corresponds to an extremum of $I[u]$.

(b) Deduce all boundary conditions which correspond to an extremum of $I[u]$ and classify them as essential or natural.

(c) Specify the space $\mathcal{U}$ of admissible solution functions for the extremization of $I[u]$ subject to homogeneous essential boundary conditions.

(d) Suggest a one-parameter polynomial approximation for a Rayleigh-Ritz solution to the boundary-value problem resulting from the extremization of $I[u]$ with homogeneous essential boundary conditions. You do not have to solve the Rayleigh-Ritz problem.