

Assignment Nr. 1

due September 9th

Problem 1

- (a) Given $\mathbf{x} \in \mathbb{R}^n$ with components (x_1, x_2, \dots, x_n) , show that the functions $\|\mathbf{x}\|_1 = \sum_{i=1}^n |x_i|$, $\|\mathbf{x}\|_2 = (\sum_{i=1}^n x_i^2)^{1/2}$ and $\|\mathbf{x}\|_\infty = \max_{1 \leq i \leq n} |x_i|$ are norms.
- (b) Determine the points of \mathbb{R}^2 for which $\|\mathbf{x}\|_1 = 1$, $\|\mathbf{x}\|_2 = 1$ and $\|\mathbf{x}\|_\infty = 1$ separately. Sketch on a single plot the geometric curves corresponding to the above points.

Problem 2

Compute the inner product $\langle u, v \rangle_{L_2(\Omega)}$ for $u = x + 1$ and $v = 3x^2 + 1$, given that $\Omega = [-1, 1]$.

Problem 3

- (a) Write the explicit form of the inner product $\langle u, v \rangle_{H^2(\Omega)}$ and the associated norm on $H^2(\Omega)$ given that $\Omega \subset \mathbb{R}^2$.
- (b) Using the result of part (a), find the “distance” in the H^2 -norm between functions $u = \sin x + y$ and $v = x$ for $\Omega = \{(x, y) \mid |x| \leq \pi, |y| \leq \pi\}$.

Problem 4

Show the parallelogram law for inner product spaces:

$$\|u + v\|^2 + \|u - v\|^2 = 2\|u\|^2 + 2\|v\|^2.$$

Problem 5

Show that the integral I given by

$$I = \int_a^b \delta^2(x) dx \quad , \quad a < 0 < b$$

is not well-defined.

Hint: Use the definition of the Dirac-delta function $\delta(x)$ and construct a sequence of integrals I_n converging to I .

Problem 6

Given $\Omega = \{\mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x}\|_2 \leq 1\}$, show that the function $f(\mathbf{x}) = \|\mathbf{x}\|_2^\alpha$ belongs to the Sobolev space $H^1(\Omega)$ for $\alpha > 0$. In addition, determine the range of α so that f also belong to $H^2(\Omega)$.