Fall 2004

ME267 Final Exam

Due in Etcheverry 6121 on or before December 24, 2004
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Consider Jupiter's atmosphere and the Cartesian version of Euler's equation that has already had the small terms due to curvature (of the spherical planet) and centrifugal terms (which distort it from spherical) removed:

$$\partial \mathbf{v}_\perp/\partial t = -(\mathbf{v}_\perp \cdot \nabla)\mathbf{v}_\perp - \mathbf{v}_z(\partial \mathbf{v}_\perp/\partial z) - (\nabla_\perp P)/\rho + 2\Omega(y)\mathbf{v}_\perp \times \hat{\mathbf{z}}, \tag{1}$$

$$\partial \mathbf{v}_z/\partial t = -(\mathbf{v}_\perp \cdot \nabla)\mathbf{v}_z - \mathbf{v}_z(\partial \mathbf{v}_z/\partial z) - (\partial P/\partial z)/\rho - g, \tag{2}$$

where x and y are the local east-west and north-south coordinates, z is the local vertical coordinate, g is the local vertical component of gravity, the subscript $\perp$ refers to the horizontal directions, a "hat" above a quantity means it is a unit vector, $\mathbf{v}$ is the velocity, P the pressure, ρ the density, and $2\Omega(y)$ the local Coriolis parameter. Note that Ω is a function of y. Express the velocity and each thermodynamic variable as a sum of its "barred" and "tilded" components, where the former is the exact, steady-state and x-independent solution to eqs. (1) and (2), *e.g.*:

$$\mathbf{v} \equiv \bar{\mathbf{v}}(y) + \tilde{\mathbf{v}}. \tag{3}$$

Here we shall define $\bar{\mathbf{v}}(y) \equiv \bar{v}(y)\,\hat{\mathbf{x}}$, so that it represents the zonal, east-west winds of Jupiter whose intensities and directions vary with y. The tilded piece of the velocity represents waves, turbulence, and vortices such as the Great Red Spot. Assume $\bar{\rho}(y,z)$ and $\bar{P}(y,z)$ are functions only of y and z.

1) In general, we cannot analytically find the $\bar{\rho}(y,z)$ and $\bar{P}(y,z)$ that solve eqs. (1) and (2). However, we can find them as series expansions in the small dimensionless number $\epsilon^2 \equiv 2[\Omega][\bar{\mathbf{v}}][\bar{\rho}]L/[\bar{P}] \ll 1$, where L is the horizontal length scale and where brackets [ ] mean the characteristic value of the quantity between them. Show that for an ideal isothermal gas, the first terms in the series expansions are

$$\bar{\rho} = \rho_0 e^{-z/H} \tag{4}$$

$$\bar{P} = P_0 e^{-z/H} + gh(y)\bar{\rho}, \tag{5}$$

where ρ_0, P_0 and H are constants (and H is defined to be the vertical scale height). That is, re-write eqs. (1) and (2) in their simplified form when all of the tilded quantities are set to zero and where you explicitly use the fact that the barred quantities are independent of t and x. (Also use the fact that $(\bar{\mathbf{v}}\cdot\nabla)\bar{\mathbf{v}} = 0$, why is that true?). Now, show that when eqs. (4) and (5) are substituted into these simplified equations that these equations are either satisfied exactly or have fractional errors of

$O(\epsilon^2)$. Using these substitutions, find $h(y)$ in terms of Ω and any other parameters; find $[gH]$ in terms of the sound speed c_s, ϵ and any other parameters (Remember, $[c_s^2] = [P]/[\rho]$.); and find $[h]/[H]$ in terms of ϵ and any other parameters.

2) Assume the Rossby number $Ro \equiv [\tilde{\mathbf{v}}_\perp]/2[\Omega]L$ is of order unity or less, assume geostrophic scaling for $\tilde{P}$ (that is, assume $[\tilde{P}] = 2[\Omega][\tilde{\mathbf{v}}_\perp][\bar{\rho}]L)$, assume for *all* quantities that the vertical scale-height is H and the horizontal length-scale is L, assume $[\bar{\mathbf{v}}] \sim [\tilde{\mathbf{v}}_\perp]$, and assume $[\mathbf{v}_z] \sim Ro\,(H/L)^2[\tilde{\mathbf{v}}_\perp]$, and assume $(H/L) \ll 1$. Show that $[\tilde{P}]/[\bar{P}] \sim \epsilon^2$.

3) Re-write and simplify eqs. (1) and (2) by expressing all variables as sums of their barred and tilded components; subtract (or cancel out the terms) from each of the equations the terms that correspond to the balance of the "barred" terms, *i.e.* in the vertical equation subtract out the hydrostatic balance between $\bar{P}$ and gravity and in the horizontal equation cancel out the balance between the $\bar{P}$ term and the component of the Coriolis acceleration due to $\bar{\mathbf{v}}$. Be sure to expand $1/\rho$ as a series in $\tilde{\rho}/\bar{\rho}$. Non-dimensionalize the equations using: $[x] = [y] = L$, $[z] = H$, $[\tilde{\mathbf{v}}_\perp] = [\bar{\mathbf{v}}] = 2[\Omega]L$, $[\tilde{\mathbf{v}}_z] = Ro\,[\tilde{\mathbf{v}}_\perp](H/L)^2$, $[t] = L/[\tilde{\mathbf{v}}_\perp]$, $[\bar{\rho}] = \rho_0$, $[\bar{P}] = P_0$, $c_s^2 \equiv P_0/\rho_0$, and $[\tilde{P}] = 2[\Omega][\tilde{\mathbf{v}}_\perp][\bar{\rho}]L$. Note that $Ro = 1$.

4) Simplify the equation for the horizontal velocity, keeping only the leading order terms in your expansion of ϵ^2. In this step you may *assume* that $[\tilde{\rho}]/[\bar{\rho}] \leq \epsilon^2$. Also *assume* that $(H/L) \leq \epsilon^2$. Show that this dimensionless equation shows that $m = \epsilon$, where the Mach number m is defined $m \equiv [\tilde{\mathbf{v}}_\perp]/c_s$.

5) Simplify the equation for the vertical velocity, keeping only the leading order terms in your expansion of ϵ^2. In this step you may start by *assuming* that $[\tilde{\rho}]/[\bar{\rho}] \leq \epsilon^2$ and $(H/L) \leq \epsilon^2$. Show that this dimensionless equation proves consistently that $[\tilde{\rho}]/[\bar{\rho}] = \epsilon^2$. Hint: you will need to use the expression you found in part (1) for $[gH]$ in terms of the sound speed c_s, ϵ and any other parameters.

6) Assuming that $\bar{T}$ is nearly constant in space and time, the equation that controls the temperature T is

$$\partial T/\partial t = -(\mathbf{v}_\perp \cdot \nabla)T - \mathbf{v}_z(\partial T/\partial z) - (\gamma - 1)T\mathbf{v}_z/H - \tilde{T}/\tau, \tag{6}$$

where τ is constant and equal to the cooling/heating time of the gas due to radiation and γ is constant and equal to the ratio of the specific heats. Re-write each variable in terms of its "barred" and "tilded" components, and write the equation in dimensionless form to its leading order. Note that the τ terms tends to force $\tilde{T}$ to zero, while the $\mathbf{v}_z$ tends to make it non-zero. Assuming that $\tau > [t] \equiv L/[\tilde{\mathbf{v}}_\perp]$, find a scaling relationship for $\mathbf{v}_z$ in terms of γ and other parameters. Now assume that $\tau < [t] \equiv L/[\tilde{\mathbf{v}}_\perp]$. Find a new scaling relationship for $\mathbf{v}_z$.

7) BONUS. In class we found that the vortices in protoplanetary disks have $(H/L) > 1$. Jovian vortices are observed to have $(H/L) \ll 1$. Using the equations and scaling arguments that you just derived plus any material presented in class, explain why jovian vortices have $(H/L) \ll 1$.