University of California, Berkeley
ME 180, Engineering Analysis Using the Finite Element Method
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Quiz 4 : Solutions

Problem 1

a) Derive the approximation functions $\phi_I^e(\xi)$ for a master element with nodes at $\xi = -1$, $\xi = 0$, and $\xi = 1$, and show that it meets the typical criteria for approximation functions.

Solution (2 points): We derive the approximation functions based on the first typical criterion:

$$\phi_I^e = \delta_{IJ},\ J = 1, 2, \cdots, nen,$$

where *nen* is the number of nodes in the element. Based on this, the approximation functions can be defined as

$$\phi_1^e = \frac{(0-\xi)(1-\xi)}{(0-(-1))(1-(-1))} = \frac{(-\xi)(1-\xi)}{2},$$
$$\phi_2^e = \frac{(-1-\xi)(1-\xi)}{(-1-0)(1-0)} = (1+\xi)(1-\xi),$$
$$\phi_3^e = \frac{(-1-\xi)(0-\xi)}{(-1-1)(0-1)} = \frac{(1+\xi)(\xi)}{2}.$$

Now, checking the second typical criterion for shape functions yields the following:

$$\sum_{I=1}^{nen} \phi_I^e = 1,\ \Omega^e = [-1, 1]$$
$$= \frac{(-\xi)(1-\xi)}{2} + (1+\xi)(1-\xi) + \frac{(-1-\xi)(-\xi)}{2},$$
$$= \frac{-\xi+\xi^2}{2} + \frac{2-2\xi^2}{2} + \frac{\xi+\xi^2}{2},$$
$$= \frac{2}{2} = 1,$$

for all values of ξ.

Thus, the shape functions meet the typical criteria.

b) Write down the map $x = M(\xi)$ for an element in physical space with nodes at x_1, x_2, and x_3.

Solution (2 points): This is fairly straightforward, as follows:

$$
\begin{aligned}
x &= \sum_{I=1}^{nen} N_I^e x_I, \\
&= N_1^e x_1 + N_2^e x_2 + N_3^e x_3, \\
&= \frac{(-\xi)(1-\xi)}{2} x_1 + (1+\xi)(1-\xi) x_2 + \frac{(-1-\xi)(-\xi)}{2} x_3.
\end{aligned}
$$

c) A desirable property of the isoparametric mapping is that it is invertible for all points in the element. State the mathematical condition that expresses this property.

Solution (2 points): Invertability of the map over the whole element would be guaranteed if

$$
J = \det\left[\frac{\partial \mathbf{x}}{\partial \boldsymbol{\xi}}\right] \neq 0, \forall \boldsymbol{\xi},
$$

(for a system where the number of spatial dimensions > 1) and, in particular, we would like J to be strictly positive, to prevent elements from being overly distorted.

For 1D, this reduces to

$$
J = \frac{\partial x}{\partial \xi} \neq 0, \forall \xi.
$$

d) Consider an element in *physical* space with nodes 1, 2, and 3 at positions x_1, x_2, and x_3, such that $x_1 < x_2 < x_3$. What restriction must be placed on x_2, in terms of x_1, x_3, for the element to be "well-behaved"?

Solution (6 points): First, calculate an expression for J:

$$
\begin{aligned}
J &= \det\left(\frac{\partial \mathbf{x}}{\partial \boldsymbol{\xi}}\right), \\
&= \frac{\partial x}{\partial \xi}, \\
&= \frac{\partial N_1}{\partial \xi} x_1 + \frac{\partial N_2}{\partial \xi} x_2 + \frac{\partial N_3}{\partial \xi} x_3, \\
&= \left(\frac{-1+2\xi}{2}\right) x_1 + (-2\xi)\, x_2 + \left(\frac{1+2\xi}{2}\right) x_3.
\end{aligned}
$$

Next, state the problem we would like to answer. In this case, we want to know where we can place x_2 in physical space, in terms of x_1 and x_3, so that the element is well-behaved (i.e., J is positive, $\forall \xi$). To this end, let us first look at how J varies over the master element (i.e., how J varies with respect to ξ):

$$
\frac{\partial J}{\partial \xi} = x_1 - 2x_2 + x_3.
$$

So, for fixed values of x_1, x_2, and x_3, the slope is a constant, i.e., $J(\xi)$ is a straight line. Also, we note that J is continuous in the variable ξ. So, lets look at the three cases:

- The slope equals zero: This corresponds to $x_2 = x_1 + \frac{1}{2}(x_3 - x_1)$. In this case, we only need to verify that J is positive, for any value of ξ. Indeed:

$$J(\xi = 0) = \frac{x_3 - x_1}{2}.$$

  So, x_2 can be at the midpoint of the element, no problem.

- The slope is positive: This corresponds to $x_2 = x_1 + \alpha(x_3 - x_1)$, where $\alpha < \frac{1}{2}$. To ensure that J is positive everywhere, we only need to choose x_2 so that J is positive at the left endpoint of the domain. Letting $J = 0$ results in

$$J = 0 \Rightarrow x_2 = \frac{\left(-\frac{1}{2} + \xi\right) x_1 + \left(\frac{1}{2} + \xi\right) x_3}{2\xi}.$$

  Now, evaluating that at $\xi = -1$ yields

$$x_2(\xi = -1) = \frac{\left(-\frac{1}{2} - 1\right) x_1 + \left(\frac{1}{2} - 1\right) x_3}{2(-1)} = \frac{3}{4}x_1 + \frac{1}{4}x_3 = x_1 + \frac{1}{4}(x_3 - x_1).$$

  So, α cannot actually equal $\frac{1}{4}$, since that would result in $J = 0$, so the condition for this case must be $x_2 > x_1 + \frac{1}{4}(x_3 - x_1)$.

- The slope is negative: This corresponds to $x_2 = x_1 + \alpha(x_3 - x_1)$, where $\alpha > \frac{1}{2}$. Now, to ensure that J is positive everywhere, we only need to choose x_2 so that J is positive at the right endpoint of the domain. A similar calculation as the previous case yields

$$x_2(\xi = 1) = \frac{\left(-\frac{1}{2} + 1\right) x_1 + \left(\frac{1}{2} + 1\right) x_3}{2} = \frac{1}{4}x_1 + \frac{3}{4}x_3 = x_1 + \frac{3}{4}(x_3 - x_1).$$

  Again, in this case, α cannot actually equal $\frac{3}{4}$.

Thus, the final condition on x_2 required to make the Jacobian positive at the endpoints (and thus everywhere in the domain) is

$$x_1 + \frac{1}{4}(x_3 - x_1) < x_2 < x_1 + \frac{3}{4}(x_3 - x_1).$$

Problem 2

Consider the system described by the strong form

$$
\begin{aligned}
\frac{\partial}{\partial x}\left(D(x)\frac{\partial c}{\partial x}\right) - \tau c(x,t) = \dot{c}(x,t) &\quad \text{in } \Omega \times I = (0,L)\times(0,T], \\
c = \bar{c} &\quad \text{on } \Gamma_u \times I, \\
-D(x)\frac{\partial c}{\partial x} = \bar{q}(t) &\quad \text{on } \Gamma_q \times I, \\
c = c_0 &\quad \text{on } \Omega \times 0.
\end{aligned}
$$

a) Derive the weak form for this system.

Solution (6 points): This calculation will have an added twist: the inclusion of $\dot{c}$ in the PDE. However, at this point, we can proceed with all of the usual steps:

$$
\begin{aligned}
&\frac{\partial}{\partial x}\left(D(x)\frac{\partial c}{\partial x}\right) - \tau c(x) - \dot{c}(x,t) = 0, \\
&w(x)\left[\frac{\partial}{\partial x}\left(D(x)\frac{\partial c}{\partial x}\right) - \tau c(x,t) - \dot{c}(x,t)\right] = 0, \\
&\int_\Omega w(x)\left[\frac{\partial}{\partial x}\left(D(x)\frac{\partial c}{\partial x}\right) - \tau c(x,t) - \dot{c}(x,t)\right] d\Omega = 0, \\
&\int_\Omega w(x)\frac{\partial}{\partial x}\left(D(x)\frac{\partial c}{\partial x}\right) - \tau w(x)c(x,t) - w(x)\dot{c}(x,t)\, d\Omega = 0, \\
&\int_\Omega -\frac{\partial w}{\partial x}D(x)\frac{\partial c}{\partial x} - \tau w(x)c(x,t) - w(x)\dot{c}(x,t)\, d\Omega = -\int_\Omega \frac{\partial}{\partial x}\left(w(x)D(x)\frac{\partial c}{\partial x}\right) d\Omega, \\
&\int_\Omega \left[D(x)\frac{\partial w}{\partial x}\frac{\partial c}{\partial x} + \tau w(x)c(x,t) + w(x)\dot{c}(x,t)\right] d\Omega = \int_{\Gamma_q} w(x)D(x)\frac{\partial c}{\partial x}n_x\, d\Gamma, \\
&\int_\Omega \left[D(x)\frac{\partial w}{\partial x}\frac{\partial c}{\partial x} + \tau w(x)c(x,t) + w(x)\dot{c}(x,t)\right] d\Omega = -w(L)\bar{q}(t).
\end{aligned}
$$

Note that that last line uses the assumption that $\Gamma_q = L$.

b) Define a (simple) approximation to calculate $\dot{c}$. Use that, along with the appropriate approximations for $c(x,t)$ and $w(x)$, to calculate the system of equations. Show the matrix form of the system of equations, similar to $\mathbf{Ku} = \mathbf{f}$ for time-independent problems, and show the entries for each matrix of the equation, e.g.,

$$
\mathbf{K} = \begin{bmatrix} \int_\Omega A_1 \frac{\partial \phi_1}{\partial x}\frac{\partial \phi_1}{\partial x} d\Omega & \cdots \\ \vdots & \ddots \end{bmatrix}, \mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \end{bmatrix}, \mathbf{f} = \begin{bmatrix} \int_\Omega f(x)\phi_1\, d\Omega \\ \vdots \end{bmatrix}.
$$

To make things easier, make the following assumptions: 1) the approximation functions are defined over the whole domain (i.e., don't worry about elements), 2) the approximation functions are defined in terms of x (i.e., don't worry about parametric space), and 3) don't apply the Dirichlet BCs to the matrix system of equations.

Solution (8 points): Define a simple time integrator as

$$\dot{c}_{n+1} = \frac{c_{n+1} - c_n}{\Delta t}.$$

The typical approximations are as follows:

$$c(x,t) = \sum_{I=1}^{nnp} N_I c_I,$$
$$\dot{c}(x,t) = \sum_{I=1}^{nnp} N_I \dot{c}_I,$$
$$w(x) = \sum_{I=1}^{nnp} N_I w_I,$$

where nnp is the number of nodal points in the domain. Note that the spatial and time dependencies are split, with the approximation functions being defined as $N_I = N_I(x)$, and the nodal values of c being defined as $c_I = c_I(t)$.

So, substituting in the time integrator and the approximation for w yields the following:

$$\sum_{I=1}^{nnp} w_I \left[\int_\Omega \left[D(x) \frac{\partial N_I}{\partial x} \frac{\partial c_{n+1}}{\partial x} + \tau N_I c_{n+1} + N_I \left(\frac{c_{n+1} - c_n}{\Delta t} \right) \right] d\Omega \right] = \sum_{I=1}^{nnp} w_I \left[-N_I(L) \bar{q}_{n+1} \right],$$
$$\sum_{I=1}^{nnp} w_I \left[\int_\Omega \left[D(x) \frac{\partial N_I}{\partial x} \frac{\partial c_{n+1}}{\partial x} + \tau N_I c_{n+1} + N_I \frac{c_{n+1}}{\Delta t} \right] d\Omega \right] = \sum_{I=1}^{nnp} w_I \left[-N_I(L) \bar{q}_{n+1} + \int_\Omega N_I \frac{c_n}{\Delta t} d\Omega \right].$$

and since w is arbitrary, we have a system of equations

$$\int_\Omega \left[D(x) \frac{\partial N_I}{\partial x} \frac{\partial c_{n+1}}{\partial x} + \tau N_I c_{n+1} + N_I \frac{c_{n+1}}{\Delta t} \right] d\Omega = -N_I(L) \bar{q}_{n+1} + \int_\Omega N_I \frac{c_n}{\Delta t} d\Omega, \quad I = 1, 2, \cdots, nnp.$$

Now, substitute in the approximation for $c(x,t)$:

$$\int_\Omega \left[D(x) \frac{\partial N_I}{\partial x} \sum_{J=1}^{nnp} \frac{\partial N_J}{\partial x} c_{J_{n+1}} + \tau N_I \sum_{J=1}^{nnp} N_J c_{J_{n+1}} + N_I \frac{1}{\Delta t} \sum_{J=1}^{nnp} N_J c_{J_{n+1}} \right] d\Omega = -N_I(L) \bar{q}_{n+1} + \int_\Omega N_I \sum_{J=1}^{nnp} N_J \frac{c_{Jn}}{\Delta t} d\Omega,$$
$$I = 1, 2, \cdots, nnp.$$

So, finally, we can write out the matrix equation as

$$\mathbf{K} \mathbf{c}_{n+1} = \mathbf{f},$$

where

$$\mathbf{K} = \begin{bmatrix} \int_\Omega \left[D(x)\frac{\partial N_1}{\partial x}\frac{\partial N_1}{\partial x} + (\tau + \frac{1}{\Delta t})N_1N_1\right] d\Omega & \int_\Omega \left[D(x)\frac{\partial N_1}{\partial x}\frac{\partial N_2}{\partial x} + (\tau + \frac{1}{\Delta t})N_1N_2\right] d\Omega & \cdots & \int_\Omega \left[D(x)\frac{\partial N_1}{\partial x}\frac{\partial N_{nnp}}{\partial x} + (\tau + \frac{1}{\Delta t})N_1N_{nnp}\right] d\Omega \\ \int_\Omega \left[D(x)\frac{\partial N_2}{\partial x}\frac{\partial N_1}{\partial x} + (\tau + \frac{1}{\Delta t})N_2N_1\right] d\Omega & \int_\Omega \left[D(x)\frac{\partial N_2}{\partial x}\frac{\partial N_2}{\partial x} + (\tau + \frac{1}{\Delta t})N_2N_2\right] d\Omega & \cdots & \int_\Omega \left[D(x)\frac{\partial N_2}{\partial x}\frac{\partial N_{nnp}}{\partial x} + (\tau + \frac{1}{\Delta t})N_2N_{nnp}\right] d\Omega \\ \vdots & \vdots & \ddots & \vdots \\ \int_\Omega \left[D(x)\frac{\partial N_{nnp}}{\partial x}\frac{\partial N_1}{\partial x} + (\tau + \frac{1}{\Delta t})N_{nnp}N_1\right] d\Omega & \int_\Omega \left[D(x)\frac{\partial N_{nnp}}{\partial x}\frac{\partial N_2}{\partial x} + (\tau + \frac{1}{\Delta t})N_{nnp}N_2\right] d\Omega & \cdots & \int_\Omega \left[D(x)\frac{\partial N_{nnp}}{\partial x}\frac{\partial N_{nnp}}{\partial x} + (\tau + \frac{1}{\Delta t})N_{nnp}N_{nnp}\right] d\Omega \end{bmatrix},$$

$$\mathbf{c}_{n+1} = \begin{bmatrix} c_{1_{n+1}} \\ c_{2_{n+1}} \\ \vdots \\ c_{nnp_{n+1}} \end{bmatrix},$$

and $\mathbf{f} = \mathbf{t} + \mathbf{A}\mathbf{c}_n$, where

$$\mathbf{t} = \begin{bmatrix} -N_1(L)\bar{q}_{n+1} \\ -N_2(L)\bar{q}_{n+1} \\ \vdots \\ -N_{nnp}(L)\bar{q}_{n+1} \end{bmatrix},$$

where all entries except one will be zero, and

$$\mathbf{A} = \frac{1}{\Delta t}\begin{bmatrix} \int_\Omega N_1N_1\, d\Omega & \int_\Omega N_1N_2\, d\Omega & \cdots & \int_\Omega N_1N_{nnp}\, d\Omega \\ \int_\Omega N_2N_1\, d\Omega & \int_\Omega N_2N_2\, d\Omega & \cdots & \int_\Omega N_2N_{nnp}\, d\Omega \\ \vdots & \vdots & \ddots & \vdots \\ \int_\Omega N_{nnp}N_1\, d\Omega & \int_\Omega N_{nnp}N_2\, d\Omega & \cdots & \int_\Omega N_{nnp}N_{nnp}\, d\Omega \end{bmatrix}.$$

Problem 3

Consider a linear system of equations, given as

$$\mathbf{A}\mathbf{u} = \mathbf{b}.$$

a) Define *conjugacy* of the vectors **a** and **b** with respect to a matrix **A**.

Solution (2 points): Conjugacy of the vectors **a** and **b** with respect to the matrix **A** is defined as

$$\mathbf{a}^T\mathbf{A}\mathbf{b} = 0.$$

b) State the requirements on a linear system of equations that need to exist for the method of conjugate gradients to work, and define those requirements mathematically.

Solution (2 points): The system must be representable as a symmetric, positive-definite matrix. Symmetry is defined as

$$\mathbf{A} = \mathbf{A}^T,$$

and positive-definiteness of the matrix $\mathbf{A}$ is defined as

$$\mathbf{a}^T \mathbf{A} \mathbf{a} > 0, \forall \mathbf{a}.$$

There are various other sufficient conditions for positive definiteness. The other one most often stated was that

$$\min(\lambda_i(\mathbf{A})) > 0,$$

where λ_i are the eigenvalues of $\mathbf{A}$, which implies that *all* of the eigenvalues of $\mathbf{A}$ are positive.

c) Show that the solution to an appropriate system of equations is the minimizer of its *quadratic form*, or *potential*.

Solution (4 points): The quadratic form (or potential) of a system of equations can be defined as

$$f(x) = \frac{1}{2}\mathbf{x}^T \mathbf{A} \mathbf{x} - \mathbf{x}^T \mathbf{b}.$$

The minimizer of this potential can be found by setting the derivative equal to zero, as follows:

$$\begin{aligned} \text{grad}\, f = \mathbf{0} = \frac{\partial f}{\partial \mathbf{x}} &= \frac{1}{2}\mathbf{A}^T \mathbf{x} + \frac{1}{2}\mathbf{A}\mathbf{x} - \mathbf{b}, \\ &= \mathbf{A}\mathbf{x} - \mathbf{b}, \text{ (for symmetric } \mathbf{A}) \end{aligned}$$

which is the definition of the solution to the system of equations.

d) Given the system

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \mathbf{u}^{(1)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

derive the conjugate gradient method via example (i.e., show where all of the numbers go). By *derive*, I mean 1) define each necessary step, and 2) describe steps whose rationale or purpose is not obvious. Perform two complete iterations, with a final calculation of $\mathbf{u}^{(3)}$.

Note: Carry around any fractions, i.e., do not try to convert fractions to decimals.

Solution (6 points): Note that, during the quiz, is was stated to ignore preconditioning. Now, start by defining the *residual* as

$$\mathbf{r}^{(i)} = \mathbf{b} - \mathbf{A}\mathbf{u}^{(i)},$$

$$\mathbf{r}^{(1)} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix},$$

and the iteration to calculate the solution as

$$\mathbf{u}^{(i+1)} = \mathbf{u}^{(i)} + \lambda^{(i)}\mathbf{z}^{(i)}.$$

To be able to iterate for $\mathbf{u}$, we need to know λ and $\mathbf{z}$.

Lets define $\mathbf{z}$ as follows:

$$\mathbf{z}^{(1)} = \mathbf{r}^{(1)} = \begin{bmatrix} 0 \\ 3 \end{bmatrix},$$

$$\mathbf{z}^{(i)} = \mathbf{r}^{(i)} + \theta^{(i)}\mathbf{z}^{(i-1)}.$$

Even though we don't need it yet, lets define θ so that $\mathbf{z}^{(i)}$ is $\mathbf{A}$-conjugate to $\mathbf{z}^{(i-1)}$, which results in

$$\theta^{(i)} = -\frac{\mathbf{r}^{(i)^T}\mathbf{A}\mathbf{z}^{(i-1)}}{\mathbf{z}^{(i-1)^T}\mathbf{A}\mathbf{z}^{(i-1)}}.$$

Choosing λ to minimize the potential energy results in

$$\lambda^{(i)} = \frac{\mathbf{z}^{(i)^T}\mathbf{r}^{(i)}}{\mathbf{z}^{(i)^T}\mathbf{A}\mathbf{z}^{(i)}},$$

$$\lambda^{(1)} = \frac{\begin{bmatrix} 0 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix}}{\begin{bmatrix} 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix}} = \frac{1}{2}.$$

Thus,

$$\mathbf{u}^{(2)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{3}{2} \end{bmatrix}.$$

Now, lets do one more iteration:

- Calculate $\mathbf{r}^{(2)}$:

$$\mathbf{r}^{(2)} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{3}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 0 \end{bmatrix}$$

- Calculate $\theta^{(2)}$:

$$\theta^{(2)} = -\frac{\begin{bmatrix} \frac{3}{2} & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 3 \end{bmatrix}}{\begin{bmatrix} 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 3 \end{bmatrix}} = \frac{1}{4}$$

- Calculate $\mathbf{z}^{(2)}$:

$$\mathbf{z}^{(2)} = \begin{bmatrix} \frac{3}{2} \\ 0 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{3}{4} \end{bmatrix}$$

- Calculate $\lambda^{(2)}$:

$$\lambda^{(2)} = \frac{\begin{bmatrix} \frac{3}{2} & \frac{3}{4} \end{bmatrix}\begin{bmatrix} \frac{3}{2} \\ 0 \end{bmatrix}}{\begin{bmatrix} \frac{3}{2} & \frac{3}{4} \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} \frac{3}{2} \\ \frac{3}{4} \end{bmatrix}} = \frac{2}{3}$$

- Finally, update $\mathbf{u}$:

$$\mathbf{u}^{(3)} = \begin{bmatrix} 0 \\ \frac{3}{2} \end{bmatrix} + \frac{2}{3}\begin{bmatrix} \frac{3}{2} \\ \frac{3}{4} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$