ME 170: Engineering Mechanics III
Department of Mechanical Engineering
University of California at Berkeley

Fall Semester 2006

Professor Oliver M. O'Reilly

Second Midterm Examination
Thursday November 16 2006
Answer Two of the Three Questions
Two Hour Take-Home Examination

Notes

(i) Your completed exam is due in my office by 9:30am on Friday November 17 2006. Please place your exam in an envelope under my door if I am not present.

(ii) You are not allowed to discuss the contents of the exam until after Friday at 9:30am.

(iii) The exam is open books and notes. I recommend that you have copies of the Reader for this class at hand and you should freely consult them.

(iv) The exam should be completed in a single *continuous* two hour sitting.

(v) You are not permitted to look at the exam prior to, or while, studying for it.

Helpful Sections and Comments

(i) For Questions 1 and 3, the 3-2-1 and 3-1-3 sets of Euler angles are discussed on pages 177–185 of your Reader.

(ii) Inertia tensors and angular momenta are discussed on pages 203–209 of your reader.

(iii) Kinetic energy is discussed on page 209.

(iv) The solutions for all three questions of this entire exam are contained on 5 written pages.

Question 1
The Motion of a Top
(25 Points)

As shown in Figure 1, a Poisson top is a axisymmetric body with a sharp apex which is free to move on a flat surface. The material point of the top in contact with the surface is labeled P. The position vectors of this point relative to the fixed point O in the present and reference configurations are denoted by $\mathbf{x}_P$ and $\mathbf{X}_P$, respectively. The material point corresponding to the center of mass of the top is denoted by $\bar{X}$, and the position vectors of $\bar{X}$ relative to the fixed point O in the present and reference configurations are denoted by $\bar{\mathbf{x}}$ and $\bar{\mathbf{X}}$, respectively.

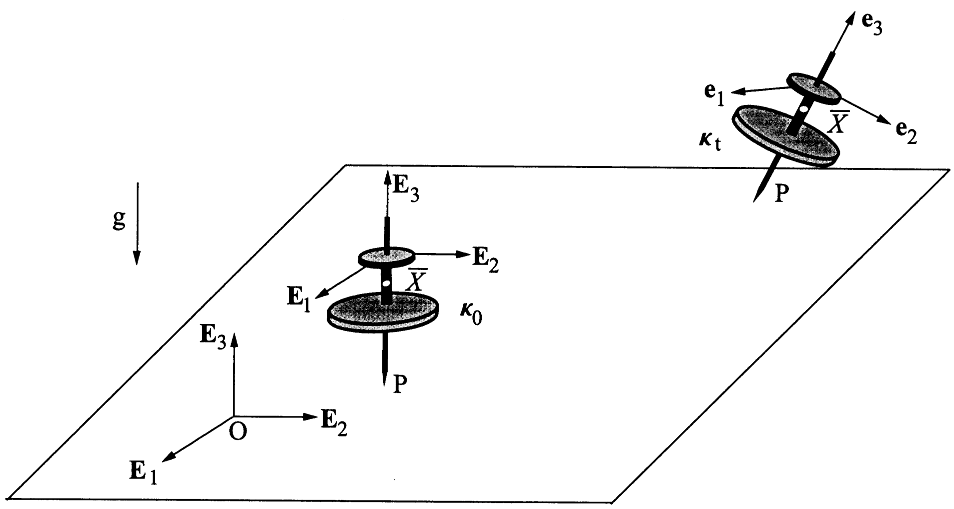

Figure 1: The fixed reference κ_0 and present κ_t configurations of a Poisson top. The surface that the top moves on is taken to be the $\mathbf{E}_1 - \mathbf{E}_2$ plane.

(a) *(7 Points)* A set of 3-2-1 Euler angles are used to describe the orientation of the top. Denoting the first angle by ψ, the second by θ and the third by ϕ, give sketches showing how these angles describe the orientation of the top.

(b) *(3 Points)* For which orientations of the top does the set of 3-2-1 Euler angles have singularities? Now, suppose a set of 3-1-3 Euler angles were used to parameterize the rotation tensor of the top, For which orientations of the top does the 3-1-3 set have singularities?

(c) *(7 Points)* Starting from the result that for any point X on a rigid body,

$$\mathbf{x} = \mathbf{Q}\mathbf{X} + \mathbf{d}, \tag{1}$$

show that

$$\mathbf{x} = \mathbf{Q}\left(\mathbf{X} - \mathbf{X}_P\right) + \mathbf{x}_P, \qquad \dot{\mathbf{x}} = \boldsymbol{\omega} \times \left(\mathbf{x} - \mathbf{x}_P\right) + \dot{\mathbf{x}}_P. \tag{2}$$

(d) *(8 Points)* The moment of inertia tensor of the top in its reference configuration has the representation

$$\mathbf{J}_O = \lambda_a \mathbf{E}_3 \otimes \mathbf{E}_3 + \lambda_t \left(\mathbf{I} - \mathbf{E}_3 \otimes \mathbf{E}_3\right). \tag{3}$$

If

$$\mathbf{\Pi}_P = \mathbf{X}_P - \bar{\mathbf{X}} = -L_3\mathbf{E}_3, \qquad \boldsymbol{\omega} = \omega_1\mathbf{e}_1 + \omega_2\mathbf{e}_2 + \omega_3\mathbf{e}_3, \tag{4}$$

then show that the angular momentum vector of the top relative to the point P is

$$\mathbf{H}_P = \lambda_a\omega_3\mathbf{e}_3 + \left(\lambda_t + mL_3^2\right)\left(\omega_1\mathbf{e}_1 + \omega_2\mathbf{e}_2\right) + L_3\mathbf{e}_3 \times m\dot{\mathbf{x}}_P. \tag{5}$$

Question 2
A Rotating Parallelepiped
(25 Points)

Consider the rectangular parallelepiped of mass m and dimensions a, b and c shown in Figure 2. Relative to the principal axes $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3\}$, the inertia tensor of this rigid body has the representation

$$\mathbf{J}_O = \sum_{i=1}^{3} \lambda_i \mathbf{A}_i \otimes \mathbf{A}_i, \tag{6}$$

where

$$\lambda_1 = \frac{m}{12}\left(b^2 + c^2\right), \qquad \lambda_2 = \frac{m}{12}\left(a^2 + c^2\right), \qquad \lambda_3 = \frac{m}{12}\left(a^2 + b^2\right). \tag{7}$$

As shown in Figure 2, the right-handed set of basis vectors $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ are oriented so that $\mathbf{E}_3$ passes diagonally through the parallelepiped. That is, this vector is parallel to the line connecting the center of mass $\bar{X}$ and the point P:

$$\mathbf{E}_3 = \frac{1}{\sqrt{a^2 + b^2 + c^2}}\left(a\mathbf{A}_1 - b\mathbf{A}_2 + c\mathbf{A}_3\right). \tag{8}$$

In addition,

$$\mathbf{E}_1 = \cos(\alpha)\mathbf{A}_1 - \sin(\alpha)\mathbf{A}_3, \tag{9}$$

where

$$\alpha = \tan^{-1}\left(\frac{a}{c}\right). \tag{10}$$

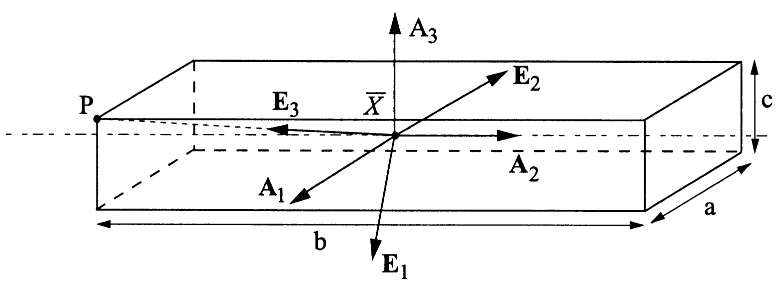

Figure 2: Schematic of parallelepiped.

(a) *(10 Points)* Show that the transformation from the basis $\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3\}$ to the basis $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ can be written in the form

$$\begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix} = \mathsf{R} \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \mathbf{A}_3 \end{bmatrix}, \tag{11}$$

where

$$\mathsf{R} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\beta) & \sin(\beta) \\ 0 & -\sin(\beta) & \cos(\beta) \end{bmatrix} \begin{bmatrix} \cos(\alpha) & 0 & -\sin(\alpha) \\ 0 & 1 & 0 \\ \sin(\alpha) & 0 & \cos(\alpha) \end{bmatrix}, \tag{12}$$

and

$$\cos(\beta) = \frac{\sqrt{a^2 + c^2}}{\sqrt{a^2 + b^2 + c^2}}, \qquad \sin(\beta) = \frac{b}{\sqrt{a^2 + b^2 + c^2}}. \tag{13}$$

(b) (*8 Points*) Show that the components $J_{0ik} = (\mathbf{J}_0\mathbf{E}_k) \cdot \mathbf{E}_i$ are

$$J_{0ik} = \sum_{r=1}^{3} R_{ir}\lambda_r R_{kr}, \tag{14}$$

You may wish to note that in matrix form this equation is

$$\begin{bmatrix} J_{011} & J_{012} & J_{013} \\ J_{012} & J_{022} & J_{023} \\ J_{013} & J_{023} & J_{033} \end{bmatrix} = \mathsf{R} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \mathsf{R}^T. \tag{15}$$

(c) (*7 Points*) If the body is given an angular velocity $\boldsymbol{\omega} = \omega\mathbf{e}_3$ where $\mathbf{e}_i = \mathbf{Q}\mathbf{E}_i$, then establish expressions for the angular momentum $\mathbf{H}$ of the body relative to its center of mass and the rotational kinetic energy T_{rot} of the body. How do these expressions simplify if $a = b = c$?

Question 3
A Grinding Machine
(25 Points)

A schematic of a grain milling machine is shown in Figure 3. The grain to be milled is placed in the bin and a roller is designed to roll without slipping on the side of the grain bin. The normal force generated by the roller is substantial and crushes the grains. The motion of the roller is achieved by two drive shafts. The first shaft (drive shaft I) has an angular velocity vector $\boldsymbol{\omega}_I = \dot{\psi}\mathbf{E}_3$. It is coupled by a universal joint at U to the second shaft. The roller is attached to the second shaft (drive shaft II) by a joint which allows it to have a rotation relative to the second shaft in the direction of $\mathbf{e}_3$. The basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ corotates with the roller.

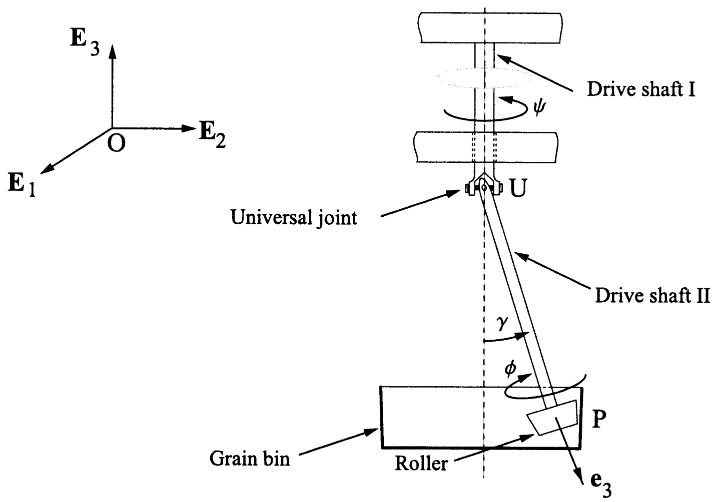

Figure 3: Schematic of a grain milling machine.

(a) *(6 Points)* Using a set of 3-1-3 Euler angles, show that the angular velocity vector of the roller has the representation

$$\begin{aligned}\boldsymbol{\omega} &= \dot{\psi}\left(\sin(\phi)\sin(\pi-\gamma)\mathbf{e}_1 + \cos(\phi)\sin(\pi-\gamma)\mathbf{e}_2 + \cos(\pi-\gamma)\mathbf{e}_3\right) \\ &\quad -\dot{\gamma}\left(\cos(\phi)\mathbf{e}_1 - \sin(\phi)\mathbf{e}_2\right) + \dot{\phi}\mathbf{e}_3. \qquad (16)\end{aligned}$$

For which orientations of the roller, does this set of Euler angles have singularities? What is the angular velocity vector $\boldsymbol{\omega}_{II}$ of drive shaft II, and what is the angular velocity of the roller relative to drive shaft II?

(b) *(6 Points)* The center of mass $\bar{X}$ of the roller has a position vector relative to the fixed point U of

$$\bar{\mathbf{x}} = H\mathbf{e}_3. \qquad (17)$$

Establish an expression for the velocity vector $\bar{\mathbf{v}}$ of the point $\bar{X}$.

(c) *(6 Points)* The instantaneous point P of contact of the roller with grain bin has the following position vector relative to $\bar{X}$:

$$\boldsymbol{\pi}_P = -r\left(\cos(\phi)\mathbf{e}_2 + \sin(\phi)\mathbf{e}_1\right). \qquad (18)$$

Using the identity $\mathbf{v}_P = \boldsymbol{\omega} \times \boldsymbol{\pi}_P + \bar{\mathbf{v}}$, establish expressions for the components $\mathbf{v}_P \cdot \mathbf{e}_i$.

(d) *(7 Points)* If $\mathbf{v}_P = \mathbf{0}$, then show that $\dot{\gamma} = 0$ and establish the relationship between $\dot{\phi}$ and $\dot{\psi}$.