

First Midterm Examination  
Tuesday October 3 2006  
Closed Books and Closed Notes  
Answer Any Two of the Three Questions

Question 1

*A Particle Subject to Two Constraints*  
25 Points

Suppose that the motion of a particle of mass  $m$  is subject to the following constraints:

$$(x\mathbf{E}_3 + \mathbf{E}_2) \cdot \mathbf{v} = 0, \quad (\mathbf{E}_2) \cdot \mathbf{v} + h(t) = 0. \quad (1)$$

(a) (5 Points) Show that one of the constraints is integrable. In addition, for this integrable constraint, specify the function  $\Psi(\mathbf{r}, t) = 0$ .

(b) (5 Points) Give a prescription for the constraint force associated with the constraints.

(c) (5 Points) Suppose that, in addition to the constraint forces, a gravitational force  $-mg\mathbf{E}_3$  acts on the particle. With the help of the balance of linear momentum  $\mathbf{F} = m\mathbf{a}$ , specify the equations governing the motion of the particle and the constraint forces.

(d) (5 Points) With the help of the work-energy theorem  $\dot{T} = \mathbf{F} \cdot \mathbf{v}$ , prove that the total energy of the particle is not conserved. Give a physical interpretation for this lack of conservation.

(e) (5 Points) Using the results from (c), how could you determine the motion of the particle and the constraint force  $\mathbf{F}_c$ ? [Hint, it may help to notice that the linear momentum of the particle in the  $\mathbf{E}_1$  direction is conserved.]

## Question 2

### *A Particle Moving on a Sphere*

25 Points

Consider a particle resting at the top of a sphere. If the particle is given a sufficiently large initial velocity, it will lose contact with the sphere. The purpose of this question is to determine this velocity.

**Background:** Several kinematical results for a spherical polar coordinate system are presented on page 4.

(a) (2 Points) Suppose the particle is in motion on the outer surface of the smooth sphere of radius  $R_0$ . A vertical gravitational force  $-mg\mathbf{E}_3$  is assumed to act on the particle. Establish an expression for the total energy  $E$  of the particle.

(b) (3 Points) Give a prescription for the constraint force  $\mathbf{F}_c$  acting on the particle.

(c) (10 Points) Show that the equations governing the motion of the particle when it is in contact with the sphere are

$$\begin{aligned} mR_0(\ddot{\phi} - \sin(\phi)\cos(\phi)\dot{\theta}^2) &= mg\sin(\phi), \\ \frac{d}{dt}(mR_0^2\sin^2(\phi)\dot{\theta}) &= 0. \end{aligned} \quad (2)$$

In addition, show that the normal force exerted by the surface of the sphere on the particle is

$$\mathbf{N} = \left(-mR_0(\dot{\phi}^2 + \sin^2(\phi)\dot{\theta}^2) + mg\cos(\phi)\right)\mathbf{e}_R. \quad (3)$$

(d) (5 Points) Using the fact that the total energy  $E$  of the particle is conserved, show that the criterion for the particle to remain on the surface of the sphere is

$$mg(3\cos(\phi) - 2\cos(\phi_0)) - mR_0(\dot{\phi}_0^2 + \sin^2(\phi_0)\dot{\theta}_0^2) > 0. \quad (4)$$

where  $\phi_0$  is the value of the initial  $\phi$  coordinate of the particle and  $\dot{\theta}_0$  and  $\dot{\phi}_0$  are the initial velocities.

(e) (5 Points) Suppose the particle is placed on top of the sphere. If the particle is given an initial speed  $v_0 > \sqrt{gR_0}$  show that it will immediately lose contact with the sphere.

**Question 3**  
*A Roller Coaster Problem*  
 25 Points

A corkscrew roller coaster can be modelled as a particle of mass  $m$  placed on a helical space curve (see Figure 1). The particle is subject to a gravitational force  $-mg\mathbf{E}_2$ .

**Background:** Assuming that  $\dot{s} = R_0\sqrt{1 + \alpha^2}\dot{\theta}$ , the Frenet triad for this space curve has the representations:

$$\mathbf{e}_t = \frac{1}{\sqrt{1 + \alpha^2}} (\mathbf{e}_\theta + \alpha\mathbf{E}_3), \quad \mathbf{e}_n = -\mathbf{e}_r, \quad \mathbf{e}_b = \frac{1}{\sqrt{1 + \alpha^2}} (\mathbf{E}_3 - \alpha\mathbf{e}_\theta). \quad (5)$$

In addition, the curvature is  $\kappa = \frac{1}{R_0(1 + \alpha^2)}$  and the torsion is  $\tau = \alpha\kappa$ .

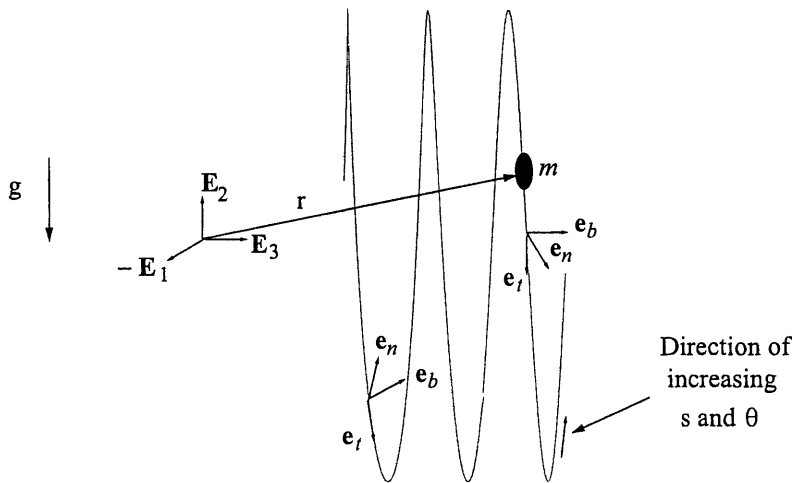


Figure 1: A particle of mass  $m$  moving on a helix.

(a) (5 Points) Starting from the representation  $\mathbf{r} = R_0\mathbf{e}_r + \alpha R_0\theta\mathbf{E}_3$ , establish expressions for the velocity and acceleration vectors of the particle. Here,  $R_0$  and  $\alpha$  are constants.

(b) (5 Points) Assuming that friction is negligible, give a prescription for the constraint force acting on the particle.

(c) (5 Points) Using the balance of linear momentum,  $\mathbf{F} = m\mathbf{a}$ , show that the equation governing the motion of the particle is

$$mR_0\sqrt{1 + \alpha^2}\ddot{\theta} = -\frac{mg}{\sqrt{1 + \alpha^2}} \cos(\theta). \quad (6)$$

(d) (5 Points) Using the work-energy theorem, prove that the total energy  $E$  of the particle is conserved. In your solution, give an expression for  $E$ .

(e) (5 Points) If you have found  $\theta(t)$  for the motions of a planar pendulum, why is it possible to directly infer the motions of the roller coaster?

## Notes on Spherical Polar Coordinates

Recall that the spherical polar coordinates  $\{R, \phi, \theta\}$  are defined using Cartesian coordinates  $\{x = x_1, y = x_2, z = x_3\}$  by the relations:

$$R = \sqrt{x_1^2 + x_2^2 + x_3^2}, \quad \theta = \arctan\left(\frac{x_2}{x_1}\right), \quad \phi = \arctan\left(\frac{\sqrt{x_1^2 + x_2^2}}{x_3}\right).$$

In addition, it is convenient to define the following orthonormal basis vectors:

$$\begin{bmatrix} \mathbf{e}_R \\ \mathbf{e}_\phi \\ \mathbf{e}_\theta \end{bmatrix} = \begin{bmatrix} \cos(\theta) \sin(\phi) & \sin(\theta) \sin(\phi) & \cos(\phi) \\ \cos(\theta) \cos(\phi) & \sin(\theta) \cos(\phi) & -\sin(\phi) \\ -\sin(\theta) & \cos(\theta) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}.$$

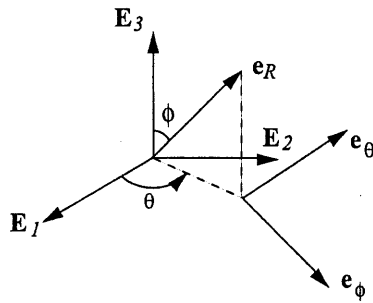


Figure 2: Spherical polar coordinates

For a particle of mass  $m$  which is unconstrained, the linear momentum  $\mathbf{G}$ , kinetic energy  $T$ , and acceleration vector  $\mathbf{a}$  of the particle are

$$\begin{aligned} \mathbf{G} &= m\dot{R}\mathbf{e}_R + mR\dot{\phi}\mathbf{e}_\phi + mR\sin(\phi)\dot{\theta}\mathbf{e}_\theta, \\ T &= \frac{m}{2} \left( \dot{R}^2 + R^2\dot{\phi}^2 + R^2\sin^2(\phi)\dot{\theta}^2 \right), \\ \mathbf{a} &= (\ddot{R} - R\dot{\phi}^2 - R\sin^2(\phi)\dot{\theta}^2)\mathbf{e}_R + (R\ddot{\phi} + 2\dot{R}\dot{\phi} - R\sin(\phi)\cos(\phi)\dot{\theta}^2)\mathbf{e}_\phi \\ &\quad + (R\sin(\phi)\ddot{\theta} + 2\dot{R}\dot{\theta}\sin(\phi) + 2R\dot{\theta}\dot{\phi}\cos(\phi))\mathbf{e}_\theta. \end{aligned} \tag{7}$$