Homework No. 7
Due Thursday December 7 2006

Announcements

This homework is optional and the marks from it can be used to make up points on previous homeworks. If you aren't planning on submitting this homework, I highly recommend that you look carefully at the three questions.

Exercises on Rigid Body Kinetics

As usual, these exercises are intended to help you test *parts* of your knowledge on material that is either covered in lecture or that I expect you to know from other courses. These exercises will **not** be graded and should **not** be submitted with your solution to Questions 1, 2 and 3. These Exercises, which are from Chapters 8 and 9 of your reader, will be covered in part during the Discussion Sessions.

Exercise I *Forces and Moments*

A body is in motion under the action of a single force $\mathbf{L}$. This force acts at a point X_L whose position vector relative to the center of mass of the body is $\boldsymbol{\pi}_L$. The resultant moment and force acting on the rigid body are

$$\mathbf{F} = \mathbf{L}, \qquad \mathbf{M} = \boldsymbol{\pi}_L \times \mathbf{L}. \tag{1}$$

(i) Show that the mechanical power of $\mathbf{L}$ has two equivalent representations:

$$\mathbf{L} \cdot \dot{\mathbf{x}}_L = \mathbf{L} \cdot \bar{\mathbf{v}} + \mathbf{M} \cdot \boldsymbol{\omega}. \tag{2}$$

(ii) If $\mathbf{L}$ is conservative then

$$\mathbf{L} = -\frac{\partial U}{\partial \mathbf{x}_L} \tag{3}$$

where the potential energy $U = U(\mathbf{x}_L)$. Argue that

$$U(\mathbf{x}_L) = \hat{U}(\bar{\mathbf{x}}, \mathbf{Q}). \tag{4}$$

(iii) What are the conservative force and moment (relative to the center of mass) associated with the potential energy U?

(iv) Using the work-energy theorem,

$$\dot{T} = \mathbf{F} \cdot \bar{\mathbf{v}} + \mathbf{M} \cdot \boldsymbol{\omega} = \sum_{K=1}^{N} \mathbf{F}_K \cdot \mathbf{v}_K + \mathbf{M}_P \cdot \boldsymbol{\omega} \tag{5}$$

prove that the total energy of the body is conserved.

(v) Illustrate your solutions to (i)–(iv) by considering a spring force acting on the body and a gravitational force acting on the body.

Exercise II *Work-Energy Theorem*

Starting from the Koenig decomposition for the kinetic energy T of a rigid body, and using the balance laws, prove the work-energy theorem (5).

Exercise III *Body Moving about a Fixed Point*

Consider the system shown below from Homework 5. It consists of four interconnected rigid bodies: the drive shaft, the axle A, the axle B and the payload. The payload is rigidly connected to B and the rotation tensor of the payload is described by a set of 3-1-3 Euler angles.

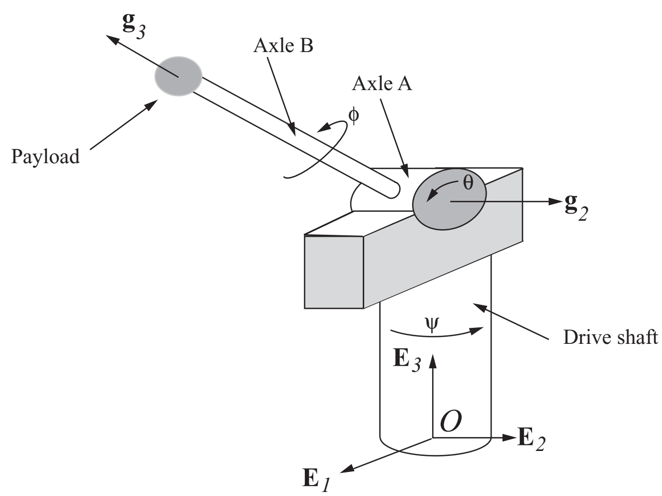

Figure 1: A robotic arm and its payload

(i) Argue that there are 6 constraints on the motion of the payload relative to B, but only 5 on the corresponding relative motions of B relative to A and A relative to the drive shaft.

(ii) Argue that there are 3 forces and 3 moments which ensure that B and the payload are rigidly connected.

(iii) Argue that there are 3 forces and 2 moments which ensure the relative motion of B and A. The moments have no component in the $\mathbf{g}^2$ direction.

(iv) Argue that there are 3 forces and 2 moments which ensure the relative motion of A and the drive shaft. The moments have no component in the $\mathbf{g}^1$ direction.

For the 3-1-3 Euler angles, the dual basis are discussed in your reader.

If the system is being driven by three motors so that ψ, θ and ϕ are prescribed functions of time, then the moments supplied by the motors are in the $\mathbf{g}^1$, $\mathbf{g}^2$, $\mathbf{g}^3$ directions!

Exercise IV *Rolling Rigid Bodies and Related Problems*

A rigid body rolls without slipping on a fixed rough surface. The instantaneous point of contact of the body with the surface is P. The position vector of this point relative to the center of mass of the rigid body is $\boldsymbol{\pi}_P$. Thus,

$$\mathbf{v}_P = \bar{\mathbf{v}} + \boldsymbol{\omega} \times \boldsymbol{\pi}_P. \tag{6}$$

We denote the combined friction force $\mathbf{F}_f$ and normal force $\mathbf{N}$ at P by $\mathbf{F}_P$:

$$\mathbf{F}_P = \mathbf{N} + \mathbf{F}_f. \tag{7}$$

(i) Why is $\mathbf{F}_P$ workless?

(ii) If a gravitational force acts on the body, then prove that the total energy of the rigid body is conserved.

(iii) What do the results of (ii) imply for a sphere rolling on a surface of revolution and for Euler's disk?

(iv) How would you apply the analysis of (ii) to the problem of a top spinning about a fixed point O?

(v) For the sphere rolling on the turntable in Question 1 below, why isn't the total energy of the sphere conserved?

Exercise V *Potential Energy of a Rigid Body*

A rigid body has a potential energy $U = U\left(\bar{\mathbf{x}}, \gamma^i\right)$, where $\gamma^1, \gamma^2, \gamma^3$ are the Euler angles used to parametrize $\mathbf{Q}$. If the conservative force $\mathbf{F}$ and conservative moment $\mathbf{M}$ are such that

$$-\dot{U} = \mathbf{F} \cdot \dot{\bar{\mathbf{x}}} + \mathbf{M} \cdot \boldsymbol{\omega}, \tag{8}$$

then verify that

$$\mathbf{F} = -\sum_{i=1}^{3} \frac{\partial U}{\partial x_i} \mathbf{E}_i, \quad \mathbf{M} = -\sum_{i=1}^{3} \frac{\partial U}{\partial \gamma^i} \mathbf{g}^i, \tag{9}$$

where $x_i = \bar{\mathbf{x}} \cdot \mathbf{E}_i$ and $\mathbf{g}^i$ are the basis vectors for the dual Euler basis: $(\boldsymbol{\omega} \cdot \mathbf{g}^i = \dot{\gamma}^i)$.

Consider a spring of stiffness K and unstretched length L which acts at a point P of a rigid body. Show that the potential energy of the spring is a function of the position vector of the center of mass and the Euler angles.

Exercise VI *A Potential Energy Function*

Suppose that a body is suspended by a torsional spring of stiffness K. The orientation of the body is prescribed by a set of 3-2-3 Euler angles, then the potential energy function is

$$U = \frac{1}{2} K \psi^2. \tag{10}$$

Show that the conservative moment $\mathbf{M}_{con}$ and force $\mathbf{F}_{con}$ due to the spring are

$$\mathbf{M}_{con} = -K\psi \mathbf{g}^1, \qquad \mathbf{F}_{con} = \mathbf{0}. \tag{11}$$

Show that $\mathbf{M}_{con}$ also depends on θ. If you have a chance, plot the vector $\mathbf{M}_{con}$ as ψ and θ are varied.

Exercise VII *Constraints*

A rigid shaft of length L is held in place by two bearings which are located at its ends in such a manner that the shaft is free to rotate about its longitudinal axis. After definining appropriate coordinate systems, describe the five constraints on the motion of the shaft? What is the system of constraint forces and moments which are needed to enforce these constraints?

Give several examples of machine components where constraints of the form you described occur.

Exercise VIII *Sliding Friction*

Suppose a body is sliding with one point P in contact with a horizontal surface. If the sole applied force on the body is gravitational, then show that the total energy of the body decreases with time:

$$\dot{E} = -\mu_k \left\|\mathbf{N}\right\| \left\|\mathbf{v}_P\right\| \leq 0. \tag{12}$$

Exercise IX *Governing Equations*

Show that the balance of angular momentum of a rigid body can be written in the form

$$\begin{aligned}
\lambda_1\dot{\omega}_1 + (\lambda_3 - \lambda_2)\omega_3\omega_2 &= \mathbf{M}\cdot\mathbf{e}_1, \\
\lambda_2\dot{\omega}_2 + (\lambda_1 - \lambda_3)\omega_3\omega_1 &= \mathbf{M}\cdot\mathbf{e}_2, \\
\lambda_3\dot{\omega}_3 + (\lambda_2 - \lambda_1)\omega_1\omega_2 &= \mathbf{M}\cdot\mathbf{e}_3.
\end{aligned} \tag{13}$$

Suppose a body is in motion and is subject to a resultant force and resultant moment

$$\mathbf{F} = -mg\mathbf{E}_3, \qquad \mathbf{M} = 10t\left\|\bar{\mathbf{v}}\right\|\mathbf{e}_3. \tag{14}$$

Outline how you would determine the motion of the rigid body.

Question 1: *Motion of a Sphere on a Turntable*

A sphere of mass m and radius R moves on a turntable. The contact between the sphere and the turntable is rough. In addition, the center O of the turntable is fixed and the turntable rotates about the vertical $\mathbf{E}_3$ with an angular speed Ω. This problem is adopted from a recent paper by Lewis and Murray[1].

(**a**) Suppose that the sphere is rolling on the turntable. The position vector of the point of contact of the sphere with the turntable is $\boldsymbol{\pi}_P = -R\mathbf{E}_3$. Show that the motion of the sphere is subject to three constraints:

$$\bar{\mathbf{v}} + \boldsymbol{\omega} \times (-R\mathbf{E}_3) = \Omega \mathbf{E}_3 \times \bar{\mathbf{x}}. \tag{15}$$

Using the representations $\boldsymbol{\omega} = \sum_{i=1}^{3} \Omega_i \mathbf{E}_i$, and $\bar{\mathbf{x}} = \sum_{i=1}^{3} x_i \mathbf{E}_i$, show that the three constraints imply that

$$\ddot{x}_1 - R\dot{\Omega}_2 + \Omega \dot{x}_2 = 0, \quad \ddot{x}_2 + R\dot{\Omega}_1 - \Omega \dot{x}_1 = 0, \quad \ddot{x}_3 = 0. \tag{16}$$

(**b**) Assuming that a vertical gravitational force acts on the sphere, draw a free–body diagram of the sphere.

(**c**) Using a balance of linear momentum and with the assistance of the constraints, show that the constraint force acting on the sphere is

$$\mathbf{F}_c = mg\mathbf{E}_3 - m\Omega(\dot{x}_2\mathbf{E}_1 - \dot{x}_1\mathbf{E}_2) + mR(\dot{\Omega}_2\mathbf{E}_1 - \dot{\Omega}_1\mathbf{E}_2). \tag{17}$$

(**d**) Using a balance of angular momentum and with the assistance of the results of (a)–(c), show that the equations governing the motion of the sphere are

$$\begin{aligned}
\ddot{x}_1 &= R\dot{\Omega}_2 - \Omega\dot{x}_2, \\
\ddot{x}_2 &= -R\dot{\Omega}_1 + \Omega\dot{x}_1, \\
\dot{x}_3 &= 0, \\
\dot{\Omega}_1 &= \left(\frac{m\Omega R}{\mu + mR^2}\right)\dot{x}_1, \\
\dot{\Omega}_2 &= \left(\frac{m\Omega R}{\mu + mR^2}\right)\dot{x}_2, \\
\dot{\Omega}_3 &= 0.
\end{aligned} \tag{18}$$

Here, $\mu = \frac{2mR^2}{5}$. Why are these equations sufficient to determine the motion $(\bar{\mathbf{x}}(t), \mathbf{Q}(t))$ of the sphere?

(**e**) For the special case where $\Omega = 0$, show that the center of mass of the sphere will move in a straight line with a constant speed, and that the angular velocity vector $\boldsymbol{\omega}$ of the sphere will be constant.

(**f**) Numerically integrate (18) for a variety of initial conditions. Is it possible for the sphere to fall off a turntable of radius R_0? In choosing your initial conditions $(\bar{\mathbf{x}}(t_0), \bar{\mathbf{v}}(t_0), \boldsymbol{\omega}(t_0))$ make sure that they are compatible with the rolling condition.

(**g**) Why isn't the total energy of the sphere conserved?

[1] A. D. Lewis and R. M. Murray, "Variational principles for constrained systems - theory and experiment," *International Journal of Non–Linear Mechanics*, **30**, (1995) 793–815.

Question 2: *Robotic Arm*

Consider once again the robotic arm shown in Figure 1. As discussed in an earlier exercise, the robotic arm has a mass m and moment of inertia tensor, relative to its center of mass,

$$\mathbf{J}_0 = \lambda_1 \mathbf{E}_1 \otimes \mathbf{E}_1 + \lambda_2 \mathbf{E}_2 \otimes \mathbf{E}_2 + \lambda_3 \mathbf{E}_3 \otimes \mathbf{E}_3. \tag{19}$$

A system of motors, which are not shown in Figure 1, are used to actuate the rotation of the robotic arm. The rotation consists of

(i) a rotation about the $\mathbf{E}_3$ axis through an angle ψ,

(ii) a rotation about $\mathbf{g}_2 = \mathbf{e}_1' = \cos(\psi)\mathbf{E}_1 + \sin(\psi)\mathbf{E}_2$ through an angle θ.

Another system of actuators prescribes the motion of the point A of the arm. The position vector of the center of mass of the arm relative to A is

$$\bar{\mathbf{x}} - \mathbf{x}_A = L\mathbf{e}_3. \tag{20}$$

(a) Interpreting the angles ψ and θ as members of a 3-1-3 set of Euler angles where $\phi = 0$, show that the dual Euler basis vectors, $\mathbf{g}^1, \mathbf{g}^2$, and $\mathbf{g}^3$, are not orthonormal. For which positions of the robot arm are these vectors not defined?

(b) Noting that the motion of the point A is prescribed: $\mathbf{x}_A = \mathbf{u}(t)$, and that the rotation of the arm is prescribed, what are the six constraints on the motion of the arm?

(c) Show that the angular momentum $\mathbf{H}$ (relative to its center of mass) of the arm is

$$\mathbf{H} = \lambda_1 \dot{\theta}\mathbf{e}_1 + \lambda_2 \dot{\psi}\sin(\theta)\mathbf{e}_2 + \lambda_3 \dot{\psi}\cos(\theta)\mathbf{e}_3. \tag{21}$$

(d) Draw a free-body diagram of the robot arm. In your free-body diagram, include a gravitational force $-mg\mathbf{E}_3$.

(e) Show that the force needed to actuate the motion of the center of mass is

$$\begin{aligned}\mathbf{F}_{act} &= mg\mathbf{E}_3 + m\ddot{\mathbf{u}} + mL(2\dot{\psi}\dot{\theta}\cos(\theta) + \ddot{\psi}\sin(\theta))\mathbf{e}_1 \\ &\quad + mL(\dot{\psi}^2\sin(\theta)\cos(\theta) - \ddot{\theta})\mathbf{e}_2 - mL(\dot{\theta}^2 + \dot{\psi}^2\sin^2(\theta))\mathbf{e}_3.\end{aligned} \tag{22}$$

In your solution, relate the force $\mathbf{F}_{act}$ to your solution for **(d)**

(f) Show the moment $\mathbf{M}_c$ needed to actuate a slewing maneuver of the robotic arm is

$$\begin{aligned}\mathbf{M}_{act} &= mL\mathbf{e}_3 \times (g\mathbf{E}_3 + \ddot{\mathbf{u}}) + 2mL^2\dot{\psi}_0\dot{\theta}_0\cos(\theta)\mathbf{e}_2 \\ &\quad - mL^2\dot{\psi}_0^2\sin(\theta)\cos(\theta)\mathbf{e}_1 + \boldsymbol{\omega} \times \mathbf{H}.\end{aligned} \tag{23}$$

During the slewing maneuver, $\dot{\theta} = \dot{\theta}_0$, $\dot{\psi} = \dot{\psi}_0$, and consequently $\boldsymbol{\omega}$ is constant. In your solution, relate the moment $\mathbf{M}_{act}$ to your solution for **(d)**.

(g) Show that the time-rate of change in the total energy E of the robotic arm is equal to the work done by $\mathbf{F}_{act}$ and $\mathbf{M}_{act}$:

$$\dot{E} = \mathbf{F}_{act} \cdot \dot{\mathbf{u}} + \mathbf{M}_{act} \cdot \dot{\psi}\mathbf{E}_3 + \mathbf{M}_{act} \cdot \dot{\theta}\mathbf{e}_1. \tag{24}$$

Question 3: *A Planetary Motion*

This famous problem is discussed in most books on satellite dynamics Consider a rigid body $\mathcal{B}$ of mass m which is in motion in a central gravitational force field about a massive fixed body of mass M. The center of this force field is assumed to be located at a fixed point O. The force, moment, and potential energy of the field are given by the approximations

$$\begin{aligned} \mathbf{F}_n &\approx m\mathbf{g}, \\ \mathbf{M}_n &\approx \left(\frac{3GM}{R^3}\right)\mathbf{c}\times(\mathbf{Jc}) = -\left(\frac{3GMm}{R^3}\right)\mathbf{c}\times(\mathbf{Ec}), \\ U_n &\approx -\frac{GMm}{R} - \left(\frac{GM}{2R^3}\right)\mathrm{tr}(\mathbf{J}) + \left(\frac{3GM}{2R^3}\right)(\mathbf{c}\cdot(\mathbf{Jc})), \end{aligned} \tag{25}$$

where $\mathbf{J}$ is the inertia tensor of $\mathcal{B}$ relative to its center of mass,

$$m\mathbf{g} = -\frac{GMm}{R^2}\mathbf{c} - \frac{3GM}{2R^4}\left(2\mathbf{J} + (\mathrm{tr}(\mathbf{J}) - 5\mathbf{c}\cdot\mathbf{Jc})\,\mathbf{I}\right)\mathbf{c}, \tag{26}$$

and

$$R = \|\bar{\mathbf{x}}\|, \qquad \mathbf{c} = \frac{\bar{\mathbf{x}}}{\|\bar{\mathbf{x}}\|}. \tag{27}$$

(**a**) Verify that $\mathbf{M}_n = -\bar{\mathbf{x}}\times\mathbf{F}_n$. What is the physical relevance of this result?

(**b**) Why are the angular momentum $\mathbf{H}_O$ and the total energy E of the satellite conserved?

(**c**) Using the balance of linear momentum, show that it is possible for the body to move in a circular orbit $\bar{\mathbf{x}} = R_0\mathbf{e}_r$ about O with a constant orbital angular velocity $\dot{\theta}_0$, which is known as the modified Kepler frequency, ω_{Km}:

$$\dot{\theta}_0 = \omega_{Km} = \omega_K\sqrt{1 + \frac{3}{2R_0^2m}(\mathrm{tr}(\mathbf{J}) - 3\mathbf{e}_r\cdot\mathbf{J}\mathbf{e}_r)}, \tag{28}$$

where the Kepler frequency was defined previously when the body of mass m was modelled as a particle:

$$\omega_K^2 = \frac{GM}{r_0^3}. \tag{29}$$

In (28), $\mathbf{e}_r = \cos(\theta)\mathbf{E}_1 + \sin(\theta)\mathbf{E}_2$, and this vector is an eigenvector of $\mathbf{J}$. That is, $\mathbf{e}_r$ is parallel to one of the principal axes of the body.

(**d**) Using the results of (**d**), show that a steady motion of the rigid body, that is, one where $\dot{\boldsymbol{\omega}} = \mathbf{0}$, is governed by the equation

$$\boldsymbol{\omega}\times(\mathbf{J}\boldsymbol{\omega}) = 3\omega_K^2\mathbf{e}_r\times(\mathbf{J}\mathbf{e}_r). \tag{30}$$

(**e**) Suppose that the body is asymmetric. That is, the principal values of $\mathbf{J}_0$ are distinct. We seek solutions of (30) such that $\boldsymbol{\omega}\cdot\mathbf{e}_r = 0$. Show that there are 6 possible solutions for $\boldsymbol{\omega}$ which satisfy (30) and 4 possible solutions for $\mathbf{e}_r$. Here, you should assume that $\mathbf{J}$ is known and as a result $\mathbf{Q}$ is known. As a result there are 6×4 possible solutions of (30).

(**f**) Suppose that the body is such that $\mathbf{J} = \mu\mathbf{I}$ where μ is a constant. Show that any constant $\boldsymbol{\omega}$ satisfies (30) and consequently, any orientation of the rigid body is possible in this case.

(**g**) Using the results of **(e)** explain why it is possible for an earth-based observer to see the same side of a satellite in a circular orbit above the Earth.