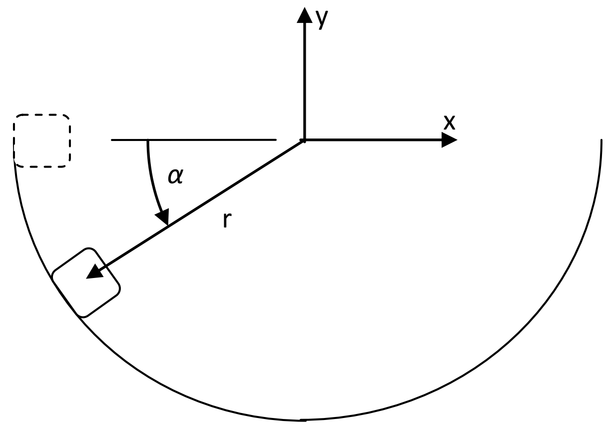
**MATLAB Problem: Half-Pipe**

In this problem you will analyze the motion of a block on a half-circle depicted in the figure. The dashed block shows the initial position. It is released from rest. The coefficient of kinetic friction is $\mu$, the mass is $m$, the circle radius is $r$, the gravitational constant is $g$.




Let $\theta$ be the angle of rotation from the Cartesian coordinate basis $(\boldsymbol{E}_x, \boldsymbol{E}_y)$ to the polar coordinate basis $(\boldsymbol{e}_r, \boldsymbol{e}_\theta)$.

1) What is the relation between $\theta$ and $\alpha$?
2) Derive the linear transformation from $(\boldsymbol{E}_x, \boldsymbol{E}_y)$ to $(\boldsymbol{e}_r, \boldsymbol{e}_\theta)$ in terms of $\alpha$.
3) Show the linear transformation derived in 2) is a rotation matrix.
4) Derive expressions for the forces acting on the block in polar coordinates, in terms of $m$, $g$, $r$, $\alpha$, $\dot{\alpha}$, $\mu$, $N$ (the normal force in the direction of $\boldsymbol{e}_r$).
5) Derive an expression for the acceleration of the block in polar coordinates using $\alpha$ (instead of $\theta$).
6) From the balance of linear momentum (combining 4) and 5)), derive an expression for $\ddot{\alpha}$ in terms of $g$, $r$, $\alpha$, $\dot{\alpha}$, $\mu$.
7) Derive the parameterization of $\ddot{\alpha}$ in the form $\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $(x_1, x_2) = (\alpha, \dot{\alpha})$.

Kinetic energy is $T = \frac{1}{2}m\mathrm{v}\cdot\mathrm{v}$, so $\dot{T} = \mathrm{F}\cdot\mathrm{v}$. Let the total power be $\dot{E} = \dot{T} + \dot{U}$. The force on the block can be broken into the sum of conservative forces P and nonconservative forces $\mathrm{P}_{nc}$. So we have $\dot{E} = (\mathrm{P} + \mathrm{P}_{nc})\cdot\mathrm{v} + \dot{U} = -\frac{\partial U}{\partial \mathrm{r}}\cdot\mathrm{v} + \mathrm{P}_{nc}\cdot\mathrm{v} + \dot{U} = -\dot{U} + \mathrm{P}_{nc}\cdot\mathrm{v} + \dot{U} = \mathrm{P}_{nc}\cdot\mathrm{v}$. Note that integrating this, we get the standard Work-Energy balance $\int_{t_1}^{t_2}\frac{d}{dt}E\,dt = E(t_2) - E(t_1) = \int_{t_1}^{t_2}\mathrm{P}_{nc}\cdot\mathrm{v}\,dt = \int_{t_1}^{t_2}\mathrm{P}_{nc}\cdot d\mathbf{r}(t)$.

8) Is the total energy $E = T + U$ of the block conserved? Derive an expression for the total energy of the block in terms of $g$, $r$, $\alpha$, $\dot{\alpha}$, $m$.
9) Derive an expression for the total power $\dot{E} = \dot{T} + \dot{U}$ of the block in terms of $g$, $r$, $\alpha$, $\dot{\alpha}$, $m$, $\mu$.
10) Extend the parameterization in 7) to include $x_3 = E$, which is the total energy of the block.

For the following parts, use values $m$ = 1 kg, $g$ = 9.81 m/s$^2$, $r$ = 1 m, $\mu$ = 0.4.

11) Write a MATLAB function that, given $(x_1, x_2, x_3)$, outputs $\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $(x_1, x_2, x_3)$ is defined as in 10). A sample function is on the website.
12) Write a MATLAB script that, using ode45 and the function from 11), simulates the motion of the block $(\alpha, \dot{\alpha})$ and the energy $T$ over the time [0, 10] seconds. A sample script is on the website.
13) Plot $\alpha, \dot{\alpha}, E$ over time. Analyze the plots. Form the plots, what can you say about $\alpha, \dot{\alpha}, E$?
14) Are linear or angular momentum conserved in particular directions? If so, which directions?