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Adaptive Rejection of Periodic Disturbances Acting on Linear Systems with Unknown Dynamics

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Abstract— **This paper proposes a novel direct adaptive control method for rejecting unknown deterministic disturbances and tracking unknown trajectories in systems with uncertain dynamics when the disturbances or trajectories are the summation of multiple sinusoids with known frequencies, such as periodic profiles or disturbances. The proposed algorithm does not require a model of the plant dynamics and does not use batches of measurements in the adaptation process. Moreover, it is applicable to both minimum and non–minimum phase plants. The algorithm is a "direct" adaptive method, in the sense that the identification of system parameters and the control design are performed simultaneously. In order to verify the effectiveness of the proposed method, an *add–on* controller is designed and implemented in the servo system of a hard disk drive to track unknown nano–scale periodic trajectories.**

I. INTRODUCTION

Control methodologies for rejecting periodic and multi–harmonic disturbances or tracking such trajectories have attracted many researchers in the past two decades. There is a multitude of applications, especially due to the dominating role of rotary actuators and power generators, that crucially rely on this type of control [1]–[4]. In this paper, we introduce a novel direct adaptive control for rejecting deterministic disturbances and tracking unknown trajectories in systems with unknown dynamics when the disturbances or trajectories are the summation of multiple sinusoids with known frequencies. Note that a periodic disturbance/trajectory with a known period can be considered as a special case of the problems under our study.

Control methods applied to this class of problems are typically categorized into two types, namely feedback methods that are based on internal model principle (IMP) [5] and feedforward algorithms that usually use an external model [6] or a *reference* signal correlated with the disturbance [7]. The classical form of internal model for periodic disturbances introduces poles on the stability boundary which can cause poor numerical properties and instability when implemented on an embedded system with finite precision arithmetic. On the other hand, feedforward control methods for this class of problems commonly estimate a set of parameters that determine the control law. The estimated parameters converge when the system is not stochastic or the adaptation gain is vanishing in a stationary (or cyclostationary) stochastic system. This implies that the estimated parameters can be frozen after convergence and the control sequence becomes a pure feedforward action that can be stored and then looked up without a need to feeding the error to the controller [3]. This is an important advantage over the feedback schemes because that type of controller should be constantly in the loop to generate the control sequence. Another advantage is that the Bode's sensitivity integral theorem does not hold true, which implies that perfect rejection can be achieved without affecting the suppression level at other frequencies. Nevertheless, analysis of the adaptive methods is, in general, more complex and relies on a set of assumptions that may not hold true in many situations.

Although rejection of sinusoidal disturbances is a classical control problem, few algorithms exist for the case that the system dynamics is unknown and possibly time–varying. Gradient descent based algorithms that use online identification schemes to obtain a finite impulse response (FIR) of the plant [8] have been proposed by both control and signal processing communities. The number of estimated parameters in these methods is usually large since low–order FIR models cannot mimic complex dynamics. The harmonic steady–state (HSS) control is another adaptive method for rejection of sinusoidal disturbances which is easy to understand and implement. However, it suffers from slow convergence since it relies on averaging over batches of data [9], [10].

This paper provides a novel direct adaptive feedforward control that does not require a model of the plant dynamics or batches of measurements. The proposed method does not rely on any assumptions about the location of plant zeros and can be applied to minimum and non–minimum phase plants. The algorithm is a "direct" adaptive method, meaning that the identification of system parameters and the control design are not separate. It will be shown that the number of adapted parameters in this scheme is significantly less than methods that identify the plant frequency response when the number of frequencies is large [10], [11].

The remainder of this paper is organized as follows. We first formalize the problem and explain the system under our study in section II. Mathematical preliminaries and notations are given in section III. The algorithm derivation is presented in section IV and simulation results for rejecting periodic disturbances in a hard disk drive nanopositioner servo system are illustrated in section V. Conclusion remarks and future

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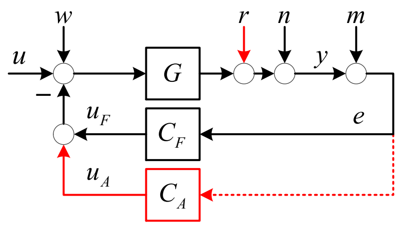

Fig. 1. Closed loop system augmented by a *plug–in* adaptive controller.

work form section VI.

II. Problem Statement

The adaptive controller proposed in this work is aimed to be implemented in a *plug–in* fashion, meaning that it is used to augment an existing robustly stable closed–loop system in order to reject periodic disturbances (track periodic trajectories) that are not well rejected (tracked) by the existing controller. In this architecture, the original controller can be designed without consideration of this special control task. To clarify this notion, we use a common Single-Input Single-Output (SISO) plant–controller interconnection shown in Fig. 1 as a running example. The blocks G and C_F in the figure respectively denote a linear time invariant (LTI) plant and an LTI feedback compensator that form a stable closed–loop sensitivity function $S := \frac{1}{1+GC_F}$. One of the main contributions of the controller that will be presented shortly is that it does not require the plant and controller dynamics. It is worth noting that this interconnection is only a running example through this paper and the proposed controller can be *plugged* to any unknown and stable LTI system regardless of its internal stabilization mechanism.

A general stochastic environment is considered for the system by appending input disturbance w, output disturbance n, and contaminating measurement noise m to our framework. Generally, the nominal feedback controller C_F is designed to compensate for these input and output noises. The special deterministic disturbance that should be compensated by the adaptive controller is denoted by r, and without loss of generality, we assume that it contaminates the plant output. Let the adaptive controller sampling time be T. The class of disturbances under our study can be written as

$$r(k) = \sum_{i=1}^{n} \alpha_i \sin\left(\omega_i kT + \delta_i\right) \tag{1}$$

where the amplitudes, α_i, and phase shifts, δ_i, are unknown but the frequencies, ω_i, are known.

Our objective is to synthesize an adaptive controller that only uses the scalar–valued error signal $e(k)$ to generate a feedforward control $u_A(k)$ such that it perfectly compensates for the effect of $r(k)$ on the error signal $e(k)$. We call it a feedforward controller because when the system dynamics and disturbance profile are time–invariant, C_A will not depend on the error signal once the control law is learned.

III. Mathematical Preliminaries and Notations

Let $R(z)$ be the transfer function from the adaptive control injection points to the error signal e, $R(z) := G(z), S(z)$ and let $A(q^{-1})$ and $B(q^{-1})$ be the polynomials that represent $R(\cdot)$ in time–domain

$$R(q^{-1}) = \frac{B(q^{-1})}{A(q^{-1})}. \tag{2}$$

We define an output disturbance, say $\bar{w}$, on $R(\cdot)$ and a polynomial $C(q^{-1})$ that satisfies

$$\frac{1}{A(q^{-1})}\left[\bar{w}(k)\right] = R\left[w(k)\right] + S\left[n(k) + m(k)\right]$$

Moreover, let $\bar{r}(k) = S\left[r(k)\right]$. When the same transfer function filters multiple input signals $i^1(k), i^2(k), \cdots, i^m(k)$, we abuse the notation and use

$$\begin{bmatrix} T[i^1(k)] \\ T[i^2(k)] \\ \vdots \\ T[i^m(k)] \end{bmatrix} = T\left[\begin{bmatrix} i^1(k) \\ i^2(k) \\ \vdots \\ i^m(k) \end{bmatrix}\right].$$

The disturbance $r(k)$ in (1) can be factorized as the inner product of a known “regressor” vector $\phi(k)$ and an unknown vector of parameters θ

$$r(k) = \theta^T \phi_R(k) \tag{3}$$

where

$$\begin{aligned} \theta^T :&= [\alpha_1 \cos(\delta_2), \alpha_1 \sin(\delta_2), \ldots \\ &\quad, \alpha_n \cos(\delta_n), \alpha_n \sin(\delta_n)] \\ \phi_R^T(k) :&= [\sin(\omega_1 kT), \cos(\omega_1 kT), \ldots \\ &\quad, \sin(\omega_n kT), \cos(\omega_n kT)]. \end{aligned}$$

Lemma 1 ([12]) *Consider $r(k)$ as a general periodic signal and $L(q^{-1})$ as a discrete-time linear system. The steady state response $\tilde{r}(k) := L(q^{-1})[r(k)]$ is periodic. Moreover, when $r(k)$ is a linear combination of sinusoidal signals factorized similarly to (3), $\tilde{r}(k)$ (in steady state) consists of sinusoidals with the same frequencies*

$$\tilde{r}(k) = L(q^{-1})\left[\theta^T \phi_R(k)\right] = \theta^T \phi_{R_L}(k)$$

where

$$\begin{aligned} \phi_{R_L}^T(k) := [&m_{L_1} \sin(\omega_1 kT + \delta_{L_1}), m_{L_1} \cos(\omega_1 kT + \delta_{L_1}), \\ &\ldots, m_{L_n} \sin(\omega_n kT + \delta_{L_n}), m_{L_n} \cos(\omega_n kT + \delta_{L_n})]. \end{aligned} \tag{4}$$

Here, m_{Li} and δ_{Li} are the magnitude and phase of $L(e^{-j\omega_i T})$ respectively. Define

$$D_{Li} := \begin{bmatrix} m_{Li}\cos(\delta_{Li}) & m_{Li}\sin(\delta_{Li}) \\ -m_{Li}\sin(\delta_{Li}) & m_{Li}\cos(\delta_{Li}) \end{bmatrix}, \tag{5}$$

$\phi_R(k)$ can be transformed to $\phi_{R_L}(k)$ by a linear transformation

$$\phi_{R_L}(k) = \underbrace{\begin{bmatrix} D_{L1} & 0 & \cdots & 0 \\ 0 & D_{L2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D_{Ln} \end{bmatrix}}_{D_L} \phi_R(k). \tag{6}$$

As a result

$$\tilde{r}(k) = L(q^{-1}) \left[\theta^T \phi_R(k)\right] = \theta^T \phi_{R_L}(k) = \theta_{\bar{R}}^T D_L \phi_R.$$

Using Lemma 1 the equivalent disturbance $\bar{r}$ can be factorized as

$$\bar{r}(k) = \theta_{\bar{R}}^T \phi_R(k) \tag{7}$$

where $\theta_{\bar{R}}$ is unknown.

IV. Adaptive Control Synthesis

A. Error Dynamics

The error sequence in time domain can be represented as a function of the closed–loop dynamics, control signals and disturbances

$$e(k) = \frac{B(q^{-1})}{A(q^{-1})} \left(u(k) + u_A(k)\right) + \frac{1}{A(q^{-1})} \bar{w}(k) + \bar{r}(k) \tag{8}$$

where $u(k)$ is an exogenous excitation signal and $\bar{w}(k)$ is an unmeasurable wide–sense stationary sequence of independent random values with finite moments. We assume that the nominal feedback controller is able to stabilize the open loop plant, i.e. $A(p)$ has all roots strictly outside the unit circle. Although a real dynamic system cannot be exactly described by finite order polynomials, in most of applications A and, B can be determined such that they give a finite vector difference equation that describes the recorded data as well as possible, i.e.

$$\begin{aligned} A(q^{-1}) &:= 1 + a_1 q^{-1} + a_2 q^{-2} + \cdots + a_{n_A} q^{-n_A} \\ B(q^{-1}) &:= b_1 q^{-1} + b_2 q^{-2} + \cdots + b_{n_B} q^{-n_B}. \end{aligned}$$

Without loss of generality, we assume that the relative degree of the transfer function from the controllable input channel to the output is 1, which implies that $n_A = n_B$. The analysis for other non-negative relative degrees is very similar to the sequel, but the notation would be more tedious due to differences in vector/matrix sizes. Let $A^*(q^{-1}) := 1 - A(q^{-1})$, then the error is given by

$$\begin{aligned} e(k) = A^*(q^{-1}) e(k) + B(q^{-1}) \left(u(k) + u_A(k)\right) \\ + \bar{w}(k) + A(q^{-1}) \bar{r}(k) \end{aligned} \tag{9}$$

This equation can be represented purely in discrete time domain in a vector form

$$e(k) = \theta_A^T \phi_e(k) + \theta_B^T \left(\phi_u(k) + \phi_{u_A}(k)\right) + \tilde{r}(k) + \bar{w}(k) \tag{10}$$

where

$$\begin{aligned} \theta_A^T &:= \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n_A} \end{bmatrix}, \\ \theta_B^T &:= \begin{bmatrix} b_1 & b_2 & \cdots, & b_{n_A} \end{bmatrix}, \\ \phi_e^T(k) &:= \begin{bmatrix} e(k-1) & e(k-2) & \cdots, & e(k-n_A) \end{bmatrix}, \\ \phi_u^T(k) &:= \begin{bmatrix} u(k-1) & u(k-2) & \cdots, & u(k-n_A) \end{bmatrix}, \\ \phi_{u_A}^T(k) &:= \begin{bmatrix} u_A(k-1) & u_A(k-2) & \cdots & u_A(k-n_A) \end{bmatrix}, \end{aligned} \tag{11}$$

and $\tilde{r}(k) := A(q^{-1})\bar{r}(k)$. Note that two regressors, denoted by $\phi_u(k)$ and $\phi_{u_A}(k)$, are considered for the excitation signal $u(k)$ and the adaptive control $u_A(k)$ separately although they could be combined into a unique regressor. The rationale behind this consideration will be explained later after (19). Since disturbance $\bar{r}(k)$ is periodic and $A(q^{-1})$ is a stable filter – i.e. it operates as an FIR filter – the response $\tilde{r}(k)$ is also periodic by Lemma 1

$$\tilde{r}(k) = A(q^{-1}) \left[\theta_{\bar{R}}^T \phi_R(k)\right] = \theta_{\bar{R}}^T \phi_{R_A}(k)$$

where

$$\phi_{R_A}(k) = \underbrace{\begin{bmatrix} D_{A1} & 0 & \cdots & 0 \\ 0 & D_{A2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D_{An} \end{bmatrix}}_{D_A} \phi_R(k). \tag{12}$$

Accordingly, $\tilde{r}(k)$ can be represented using the same regressor vector, $\phi_R(k)$

$$\tilde{r}(k) = \theta_{\bar{R}}^T \phi_{R_A}(k) = \underbrace{\theta_{\bar{R}}^T D_A}_{\theta_R^T} \phi_R(k) = \theta_R^T \phi_R(k).$$

Substituting this expression in (10) yields

$$\begin{aligned} e(k) = \theta_A^T \phi_e(k) + \theta_B^T \left(\phi_u(k) + \phi_{u_A}(k)\right) \\ + \theta_R^T \phi_R(k) + \bar{w}(k). \end{aligned} \tag{13}$$

Equation (9) shows that an ideal control signal $u_A^*(k)$ should satisfy

$$B(q^{-1}) u_A^*(k) + A(q^{-1}) \bar{r}(k) = 0. \tag{14}$$

Again, since $B(q^{-1})$ and $A(q^{-1})$ are both LTI systems and $\bar{r}(k)$ contains only sinusoidal signals, the ideal control signal $u_A^*(k)$ has to have sinusoidal contents at frequencies equal to ω_i's. This motivates us to decompose the ideal control signal into

$$u_A^*(k) = \theta_D^T \phi_R(k).$$

By this representation of the control signal, our goal will be to estimate θ_D in an adaptive manner. We define the *actual* control signal as

$$u_A(k) = \hat{\theta}_D^T(k) \phi_R(k) \tag{15}$$

where $\hat{\theta}_D(k)$ is the vector of estimated parameters that should ideally converge to θ_D. As a result, the residual in (14) when θ_D is replaced by $\hat{\theta}_D$ is

$$B(q^{-1})u_A(k) + A(q^{-1})\bar{r}(k) = B(q^{-1})\left[\hat{\theta}_D^T(k)\phi_R(k)\right] + \theta_R^T\phi_R(k). \tag{16}$$

Lemma 2 ([13]) *Let $B(q^{-1})$ have a minimal realization $B(q^{-1}) = C_B^T(qI - A_B)^{-1}B_B$. Then,*

$$B(q^{-1})\left[\hat{\theta}_D^T(k)\phi_R(k)\right] = \hat{\theta}_D^T(k)\left\{B(q^{-1})\left[\phi_R(k)\right]\right\} + w_t(k)$$

where

$$\begin{aligned} w_t(k) &:= -M(q^{-1}) \\ &\qquad \left[H(q^{-1})\left[\phi_R^T(k+1)\right]\left[\hat{\theta}(k+1) - \hat{\theta}(k)\right]\right] \\ M(q^{-1}) &:= C_B^T(qI - A)^{-1} \\ H(q^{-1}) &:= (qI - A)^{-1}B_B \end{aligned}$$

We define a new parameter vector

$$\theta_M^T(k) := \hat{\theta}_D^T(k)D_B + \theta_R^T \tag{17}$$

where D_B is a matrix similar to D_A in (6), but its block diagonal terms are formed by the magnitude and phase of $B\left(e^{-j\omega_i T}\right)$ rather than $A(e^{-j\omega_i T})$. Since the vector $\theta_M(k)$ corresponds to the imperfection in control synthesis, it is called the *residual parameters vector* throughout this section. Substituting the result of Lemma 2 in (16) yields

$$\begin{aligned} &B(q^{-1})u_A(k) + A(q^{-1})\bar{r}(k) \\ &\qquad = \hat{\theta}_D^T(k)\left(B(q^{-1})[\phi_R(k)]\right) + \theta_R^T\phi_R(k) + w_t(k) \\ &\qquad = \hat{\theta}_D^T(k)D_B\phi_R(k) + \theta_R^T\phi_R(k) + w_t(k) \\ &\qquad = \theta_M^T(k)\phi_R(k) + w_t(k) \end{aligned} \tag{18}$$

where the term $\theta_M^T(k)\phi_R(k)$ corresponds to the residual error at the compensation frequencies. The term $w_t(k)$ represents the transient excitation caused by the variation of $\hat{\theta}_D(k)$ over time. As a result, the term $\theta_B^T\phi_{u_A}(k) + \theta_R^T\phi_R(k)$ in (13) can be replaced by (18) which yields to

$$e(k) = \theta_A^T\phi_e(k) + \theta_B^T\phi_u(k) + \theta_M^T(k)\phi_R(k) + w_t(k) + \bar{w}(k). \tag{19}$$

B. Parameter Adaptation Algorithm

The error dynamics shows that the information obtained from measurements cannot be directly used to estimate $\hat{\theta}_D$ as long as the closed loop system is unknown. We propose an adaptive algorithm in this section that accomplishes the estimation of the closed loop system and control synthesis simultaneously.

Let $\hat{\theta}_A$, $\hat{\theta}_B$ and $\hat{\theta}_M$ be the estimated parameters analogous to (19). We denote the *a–priori* estimate of the error signal at time k based on the estimates at $k-1$ as

$$\hat{y}(k) = \hat{\theta}_A^T(k-1)\phi_e(k) + \hat{\theta}_B^T(k-1)\phi_u(k) + \hat{\theta}_M^T(k-1)\phi_R(k). \tag{20}$$

and accordingly, the *a–priori* estimation error is defined as

$$\epsilon^\circ(k) := e(k) - \hat{y}(k) \tag{21}$$

Assume that the estimates at time $k=0$ are initialized by either zero or some "good" values when prior knowledge about the system dynamics is available. We propose the following adaptation algorithm for updating the estimated parameters

$$\begin{aligned} \begin{bmatrix} \hat{\theta}_A(k) \\ \hat{\theta}_B(k) \\ \hdashline \hat{\theta}_M(k) \end{bmatrix} &= \begin{bmatrix} \hat{\theta}_A(k-1) \\ \hat{\theta}_B(k-1) \\ \hdashline \hat{\theta}_M(k-1) \end{bmatrix} \\ &+ \left[\begin{array}{c:c} \gamma_1(k)F^{-1}(k) & 0 \\ \hdashline 0 & \gamma_2(k)f^{-1}(k)I \end{array}\right] \begin{bmatrix} \phi_e(k) \\ \phi_u(k) \\ \hdashline \phi_R(k) \end{bmatrix} \epsilon^\circ(k). \end{aligned} \tag{22}$$

$F(k)$ is a positive (semi) definite matrix with proper dimension and $f(k)$ is a positive scalar. These step sizes can be updated via recursive least squares algorithm or least mean squares method. It is well known that the step size of adaptive algorithms in stochastic environments should converge to zero or very small values to avoid "excess error" caused by parameter variations due to noises. Therefore, positive real valued decreasing scalar sequences $\gamma_1(k)$ and $\gamma_2(k)$ are considered conjointly with the step sizes. Note that $\phi_u(k)$ should be persistently exciting of order $2n_A$ in order to guarantee that $F(k)$ is non–singular and (22) is not susceptible to numerical problems. It is clear that $f(k)$ is not subjected to this issue since $\phi_R^T(k)\phi_R(k)$ is always strictly positive.

C. Control Synthesis

Suppose that the parameter vector $\theta_M(k)$ and response matrix D_B are known at time step k. Then, a possible update rule that satisfies (14) would be

$$\begin{aligned} 0 &= \hat{\theta}_D^T(k+1)D_B + \theta_R^T \\ &= \hat{\theta}_D^T(k+1)D_B + \theta_M^T(k) - \hat{\theta}_D^T(k)D_B \Rightarrow \\ \hat{\theta}_D^T(k+1) &= \hat{\theta}_D^T(k) - \theta_M^T(k)D_B^{-1}. \end{aligned} \tag{23}$$

Here, we have used the fact that D_B is a block diagonal combination of scaled rotation matrices, which implies that it is full rank and invertible. This is an infeasible update rule since neither $\theta_M(k)$, nor D_B is known. We replace these variables by their respective estimated values and use a small step size α in order to avoid large transient and excess error

$$\hat{\theta}_D^T(k+1) = \hat{\theta}_D^T(k) - \alpha\hat{\theta}_M^T(k)\hat{D}_B^{-1}(k).$$

Note that this update rule works as a first order system that has a pole at 1. In order to robustify this difference equation we alternatively propose using a *Ridge* solution for (23). More formally, we are interested in minimizing the instantaneous cost function

$$J_c(k) := \frac{1}{2}\|\hat{\theta}_D^T(k) + \theta_R^T D_B^{-1}\|_2^2 + \frac{1}{2}\lambda\|\hat{\theta}_D^T(k)\|_2^2$$

where λ is a (positive) weight for the penalization term. We use a gradient descent algorithm to recursively update $\hat{\theta}_D$. Since the actual values of θ_M and D_B are unknown, we use the estimates. Let $\beta = 1 - \alpha\lambda$. The gradient descent update rule for $\hat{\theta}_D$ is

$$\hat{\theta}_D^T(k+1) = \beta\hat{\theta}_D^T(k) - \alpha\hat{\theta}_M^T(k)\hat{D}_B^{-1}(k). \qquad (24)$$

This expression implies that a positive value of β less than 1 results in a bounded value of $\hat{\theta}_D$ in steady state as long as $\hat{\theta}_M^T\hat{D}_B^{-1}$ stays bounded. Moreover, assuming that $\hat{\theta}_M$ and $\hat{D}_B$ converge to the actual θ_M and D_B the steady state residue is

$$\begin{aligned}\lim_{k\to\infty}\theta_M(k) &= D_B^T\lim_{k\to\infty}\hat{\theta}_D(k) + \theta_R\\ &= \frac{-\alpha}{1-\beta}\lim_{k\to\infty}\theta_M(k) + \theta_R\\ &= \frac{1-\beta}{1-\beta+\alpha}\theta_R.\end{aligned}$$

This expression shows that there is a compromise between the steady state attenuation level and robustness, and in order to achieve both, the two gains should be chosen such that

$$0 < \alpha \ll \beta < 1. \qquad (25)$$

Now that we have an update law for $\hat{\theta}_D(k)$, we have a complete algorithm for synthesizing the control signal (repeated from (15))

$$u_A(k) = \hat{\theta}_D^T(k)\phi_R(k).$$

Theorem 1 *The control update rule outlined by* (15)*,* (22) *and* (24) *make θ_M converge to $\frac{1-\beta}{1-\beta+\alpha}\theta_R$ with probability 1, the only equilibrium point of the closed loop system is stable in the sense of Lyapunov and it corresponds to $\hat{\theta}_A = \theta_A$, $\hat{\theta}_B = \theta_B$, and $\hat{\theta}_M = \frac{1-\beta}{1-\beta+\alpha}\theta_R$* ***if*** *the following conditions are satisfied:*

1) *$u(k)$ is persistently exciting of at least order $2n_A$.*
2) *$\sum_{k=1}^{\infty}\gamma_i(k) = \infty$ and $\gamma_i(k) \to 0$ as $k \to \infty$ for $i = 1, 2$.*
3) *The estimated $\hat{\theta}_A(k)$ belongs to*

$$\mathscr{D}_A := \left\{\hat{\theta}_A : 1 + \hat{a}_1 q + \cdots + \hat{a}_{n_A} q^{n_A} = 0 \Rightarrow |q| > 1\right\}.$$

infinitely often with probability one.

4) *The estimated $\hat{\theta}_B(k)$ always belongs to*

$$\begin{aligned}\mathscr{D}_B := \Big\{\hat{\theta}_B : 0 < |\hat{b}_1 e^{-j\omega_h} + \cdots + \hat{b}_{n_A} e^{-jn_A\omega_h}|\\ < \frac{\alpha}{1-\beta}|b_1 e^{-j\omega_h} + \cdots + b_{n_A} e^{-jn_A\omega_h}|,\\ \text{and } \forall h \in \{1, \ldots, n\}\Big\}.\end{aligned}$$

5) *Real $\left(\frac{B(e^{-j\omega_h})}{\hat{B}_k(e^{-j\omega_h})}\right) > 0$ for all $h \in \{1, \ldots, n\}$ infinitely often with probability one.*

Proof: Due to space limitation we only outline the sketch of the proof here. The method of analysis of stochastic recursive algorithms in [14] can be deployed to study the convergence and asymptotic behavior of the proposed algorithm. "Assumptions C" in [14] can be adopted and generalized to this algorithm with minor modifications since we are coping with a cyclostationary stochastic process. Under these regularity conditions, the results of theorem 2 in [14] imply that the only convergence point of the system is the stable equilibrium of a differential equation counterpart. This equilibrium point corresponds to the actual values of plant parameters and $\frac{1-\beta}{1-\beta+\alpha}\theta_R$. Moreover, this theorem proves that the estimated parameters converge with probability one to this equilibrium point which results in $\theta_M \to \frac{1-\beta}{1-\beta+\alpha}\theta_R$ with probability one. ■

V. Empirical Study

This section provides the experimental verification of the proposed controller. The method is used to design a plug-in controller for tracking nano-scale unknown periodic trajectories with high frequency spectra in hard disk drives (HDDs).

A so-called *single-stage* HDD uses a voice coil motor (VCM) for movements of the read/write heads [15]. The block diagram in Fig. 1 can be adopted for this mechatronic system in track-following mode. The blocks G and C_F refer to the VCM and the nominal feedback controller respectively. The signals w, r, n and m denote the (unknown) airflow disturbance, *repeatable runout* (RRO), *non-repeatable runout* (NRRO) and measurement noise respectively. The measured *position error signal* (PES) is denoted by e. The design of C_F is not discussed here, and it is assumed that this compensator can robustly stabilize the closed loop system and attenuate the broad band noises w, n and m (c.f. [16], [17]). The plug–in controller that will be explained shortly targets the RRO (r) which consists of sinusoids. An exact dynamics of the actuators is not known for each individual HDD. We do not use any information about the system dynamics or the feedback controller. In other words, everything is unknown to us except the PES. In our setup, RRO has narrow–band contents at the HDD spinning frequency ($120Hz$) and its 173 higher harmonics.

The magnitude response of the closed loop dynamics from the VCM input to the PES decays notably after $2KHz$, which makes the VCM at frequencies above $7KHz$ ineffective. Accordingly, we only focus on tracking the first 58 harmonics ($120Hz$, $240Hz$, ..., $6960Hz$) in the design of C_A in Fig. 1. The remaining 115 harmonics should be allocated to a higher bandwidth actuator which is beyond the scope of this paper and is considered as one of our future work.

The design parameters of the adaptive control algorithm are listed in Table I. In order to evaluate the convergence point of $\hat{\theta}_A$ and $\hat{\theta}_B$, we generated the transfer function $\frac{\hat{B}(q^{-1})}{\hat{A}(q^{-1})}$ that corresponds to these values. The frequency response of this (5th order) transfer function is compared to

TABLE I

PARAMETERS OF THE ADAPTIVE CONTROL ALGORITHM IN THE EMPIRICAL STUDY.

n_A (11)	α (24)	β (24)
5	4E-5	1-(2E-7)

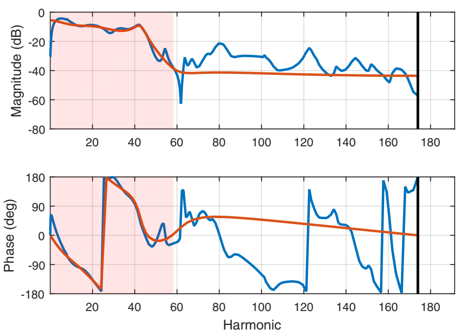

Fig. 2. Frequency response comparison of the identified model and the actual VCM loop. The shaded strip indicates the compensation frequency range.

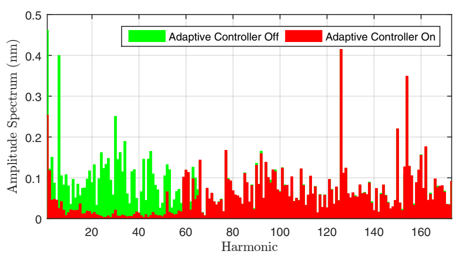

Fig. 3. Comparison of the position error spectrum before and after plugging the adaptive controller. This figure shows the amplitude of Fourier transformation only at harmonics – i.e. other frequencies are removed.

the actual transfer function of the VCM loop (a realistic 50^{th} order model) in Fig. 2. The shaded strip indicates the compensation frequency interval where the adaptive controller was active. The amplitude spectrum of the error before and after plugging the adaptive controller to the closed loop servo system are depicted in Fig. 3.

VI. CONCLUSION

A novel direct adaptive control method for the rejection of disturbances or tracking trajectories consisted of multiple sinusoidals with selective frequencies was proposed. The method is applicable to both minimum and non-minimum phase linear systems with unknown dynamics. The adapted parameters converge to the real values when a large enough excitation signal is injected to the system. In the presence of some rough knowledge about the system dynamics, the excitation signal can be reduced considerably. The analysis in this paper was performed for linear time-invariant systems. However, similar results can be extended to systems with slowly time–varying parameters.

We verified the effectiveness of the proposed control algorithm in tracking unknown nano–scale periodic trajectories in hard disk drives by designing an add–on repetitive controller that was able to track the first 58 harmonics of the disk spinning frequency. Full spectrum compensation was impossible in our running example due to the VCM limited bandwidth. This issue can be addressed by deploying a *dual–stage* mechanism that has a high–bandwidth actuator in conjunction with the VCM. Extension of the proposed method to multi–input single–output systems and experimental verification of the algorithm will form our future work.

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