How Will the Presence of Autonomous Vehicles Affect the Equilibrium State of Traffic Networks?

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***Abstract*—It is known that connected and autonomous vehicles are capable of maintaining shorter headway and distances when they form platoons of vehicles. Thus, such technologies can potentially increase the road capacities of traffic networks. Consequently, it is envisioned that their deployment will also increase the overall network mobility. In this paper, we examine the validity of this expected impact, assuming that drivers select their routes selfishly, in traffic networks with mixed vehicle autonomy, i.e. traffic networks with both regular and autonomous vehicles. We consider a nonatomic routing game on a network with inelastic (fixed) demands for a set of network O/D pairs, and study how replacing a fraction of regular vehicles by autonomous vehicles will affect the mobility of the network. Using well known US bureau of public roads (BPR) traffic delay models, we show that the resulting Wardrop equilibrium is not necessarily unique for networks with mixed autonomy. Then, we state the conditions under which the total network delay at equilibrium is guaranteed to decrease as the fraction of autonomous vehicles increases. However, we show that when these conditions do not hold, counter intuitive behaviors may occur: the total network delay can grow as the ratio of autonomous cars increases. In particular, we prove that for networks with a single O/D pair, if the road degrees of capacity asymmetry (i.e. the ratio between the road capacity when all vehicles are regular and the road capacity when all vehicles are autonomous) are homogeneous, the total delay is 1) unique, and 2) a nonincreasing continuous function of the ratio of autonomous vehicles in the network. We show that for heterogeneous degrees of capacity asymmetry, the total delay is not unique, and it can further grow as the ratio of autonomous vehicles increases. We demonstrate that similar behaviors may be observed in networks with multiple O/D pairs. We further bound such performance degradations due to the introduction of autonomy in general homogeneous networks.**

***Index Terms*—autonomous vehicles, Wardrop equilibrium, game theory, Braess's paradox, routing games, traffic networks.**

I. INTRODUCTION

Connected and autonomous vehicles technology have attracted significant attention as a result of their potentials for increasing vehicular safety and drivers' comfort. Connected technologies can be used to inform drivers about the existing hazards through vehicle to vehicle (V2V) or vehicle to infrastructure (V2I) communication. Aligned with these safety considerations, automobile companies have started to equip vehicles with autonomous capabilities. In fact, some of these capabilities, such as driver assistive technologies and adaptive cruise control (ACC) have already been deployed in vehicles.

The impact of these technologies is not limited to vehicles safety. Connected and autonomous vehicles technology can facilitate vehicle *platooning*. Vehicle platoons are groups of more than one vehicle, capable of maintaining shorter headways; thus, platooning can lead to increases in the capacities of network links [1]. Such increases can be up to three–fold [1] if all the vehicles are autonomous and connected. In addition to mobility benefits, platooning can have sustainability benefits, it can also reduce energy consumption for heavy duty vehicles [2], [3], [4].

The mobility benefits of autonomous capabilities of vehicles are not limited to increasing network capacities. There has been a focus on how to utilize vehicle autonomy and vehicle connectivity to remove signal lights from intersections and coordinate conflicting movements such that the network throughput is improved [5], [6], [7], [8]. However, in order for such approaches to be implemented, all vehicles in the network need to have autonomous capabilities. To reach the point where all vehicles are autonomous, transportation networks need to face a *transient* era, when both regular and autonomous vehicles coexist in the networks. Therefore, it is crucial to study networks with mixed autonomy.

In [9], the performance of traffic networks with mixed autonomy was studied via simulations. Moreover, it was shown in multiple works that in networks with mixed autonomy, autonomous vehicles can be utilized to stabilize the low–level dynamics of traffic networks and damp congestion shockwaves [10], [11], [12], [13], [14]. In [15] altruistic lane choice of autonomous vehicles was studied. In [16], the capacity of network links was modeled in a traffic setting with mixed autonomy. This modeling framework was further used in [17] to calculate the price of anarchy of traffic networks with mixed autonomy, where the price of anarchy is an indicator of how far the equilibrium of networks with mixed autonomy is from their social optimum that could have been achieved if a social planner had routed all vehicles. In [18], it was shown that local actions of the autonomous vehicles can lead to optimal vehicle orderings for the global network properties such as link capacities.

It is well known that due to the selfish route choice behavior of drivers, traffic networks normally operate in an equilibrium state, where no vehicle can decrease its trip time by unilaterally changing its route [19]. In this paper, we wish to study how the introduction of autonomous vehicles in the network will affect the equilibrium state of traffic networks compared to the case when all vehicles are nonautonomous. We extend our initial results presented in [20]. In particular, given a fixed demand of vehicles, we study how increasing the ratio of autnomous vehicles in the network, henceforth referred to as the *network autonomy fraction* will affect the equilibrium state of traffic

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networks. We study the system behavior when both regular and autonomous vehicles select their routes *selfishly* to investigate the necessity of centrally enforcing autonomous vehicles routing by a network manager. We state the conditions under which increasing the network autonomy fraction is guaranteed to reduce the overall network delay at equilibrium. Moreover, we show that when these conditions do not hold, counter intuitive and undesirable behaviors might occur, such as the case when increasing the portion of autonomous vehicles in the network can *increase* the overall network delay at equilibrium. Such behaviors are similar to Braess' paradox, where the construction of a new road or expanding link capacities may increase total network delay.

We model the network in a macroscopic framework where vehicle route choices are taken into account. We model the selfish route choice behavior of drivers as a nonatomic routing game [21] where drivers choose their routes selfishly until a Wardrop Equilibrium is achieved [22]. We represent a traffic network by a directed graph with a certain set of origin destination (O/D) pairs. For each O/D pair, we consider two classes of vehicles, regular and autonomous. For a given fixed demand profile along O/D pairs, we study how increasing the autonomy fraction of O/D pairs will affect the total delay of the network at equilibrium.

We first show that the equilibrium may *not* be unique. Then, we study networks with a single O/D pair and prove that if the degrees of road capacity asymmetry (i.e. the ratio between the road capacity when all vehicles are regular and the road capacity when all vehicles are autonomous) are homogeneous in the network, at equilibrium, the social or total delay of the network is unique, and further it is a monotone nonincreasing function of the network autonomy ratio. However, in networks with heterogeneous degrees of road capacity asymmetry, we first show that the social delay is not necessarily unique at equilibrium. Then, we demonstrate that, surprisingly, increasing the autonomy fraction of the network might lead to an *increase* in the overall network delay at equilibrium. This is a counter intuitive behavior as we might expect that having more autonomous vehicles in the network will always be beneficial in terms of total network delay. For networks with multiple O/D pairs, we show that similar complicated behaviors may occur, namely increasing the autonomy fraction of a single O/D pair might worsen the social delay of the network at equilibrium. Our work in fact shows that traffic paradoxes similar to the well known Braess's Paradox [23], can occur due to capacity increases provided by autonomous vehicles. We further bound such performance degradations that can arise due the presence of autonomy.

The organization of this paper is as follows. In Section II, we describe our model. We review the prior relevant results in Section III. Then, in Section IV, we study the uniqueness of equilibrium in our routing setting. Next, in Section V, we analyze mixed–autonomy networks with a single O/D pair. Subsequently, we study mixed–autonomy networks with multiple O/D pairs in Section VI. In Section VII, we discuss bounding the performance degradation that may arise due to the presence of autonomous cars. Finally, we conclude the paper and provide relevant future directions in Section VIII.

II. Nonatomic Selfish Routing

We model a traffic network by a directed graph $G = (N, L, W)$, where N and L are respectively the set of nodes and links in the network. Each link $l \in L$ in the network is a pair of distinct nodes (u, v) and represents a directed edge from u towards v. We assume that each link joins two distinct nodes; thus, no self loops are allowed. Define $W = \{(o_1, d_1), (o_2, d_2), \cdots, (o_k, d_k)\}$ to be the set of origin destination (O/D) vertex pairs of the network. A node $n \in N$ can appear in multiple O/D pairs. In a nonatomic selfish routing game, if each O/D pair has a fixed given nonzero demand, then it is called a nonatomic selfish routing game with inelastic demands. Each O/D pair consists of infinitesimally small agents where every agent decides on their path such that their own delay of travel is minimized. The delay of each path depends on how network paths are shared among different O/D pairs. For each O/D pair $w = (o_i, d_i), 1 \leq i \leq k$, we let $\mathcal{P}_w$ denote the set of all possible network paths from o_i to d_i. We assume that the network topology is such that for each O/D pair $w \in W$, there exists at least one path from its origin to its destination, i.e. $\mathcal{P}_w \neq \emptyset$. We further let $\mathcal{P} = \cup_{w \in W} \mathcal{P}_w$ denote the set of all network paths.

For an O/D pair $w \in W$, let r_w be the given fixed demand of vehicles associated with w. Furthermore, for a path $p \in \mathcal{P}_w$, let f_p be the total flow of O/D pair w along path p. Note that each path connects exactly one origin to one and only one destination; thereby, once a path is fixed, its origin and destination are uniquely determined. Consequently, there is no need to explicitly include path O/D pairs in the notation used for f_p. It is important to note that in our setting, each O/D pair w has two classes of vehicles: autonomous and regular. Consequently, for each O/D pair $w \in W$, we define α_w to be the fraction of vehicles in r_w that are autonomous. We let $r = (r_w : w \in W)$ and $\alpha = (\alpha_w : w \in W)$ be the vectors of network demand and autonomy fraction respectively. Also, for each path $p \in \mathcal{P}_w$, we use f_p^r and f_p^a to respectively denote the flow of regular and autonomous vehicles along path p. Note that for each path $p \in \mathcal{P}$, we have $f_p = f_p^r + f_p^a$. Define the network flow vector f to be a nonnegative vector of regular and autonomous flows along network paths, i.e. $f = (f_p^r, f_p^a : p \in \mathcal{P})$. A flow vector f is called feasible for a given network G, if for each O/D pair $w \in W$,

$$\sum_{p \in \mathcal{P}_w} f_p^r = (1 - \alpha_w) r_w, \text{ and } \sum_{p \in \mathcal{P}_w} f_p^a = \alpha_w r_w, \tag{1a}$$

$$f_p^r \geq 0, \text{ and } f_p^a \geq 0, \; \forall p \in \mathcal{P}_w. \tag{1b}$$

For each link $l \in L$, f_l is the total flow of vehicles along link l, i.e. $f_l = \sum_{p \in \mathcal{P}: l \in p} f_p$. Since we need to decompose the total link flow into regular and autonomous vehicles, we let f_l^r and f_l^a be the total flow of regular and autonomous vehicles along link l respectively. In fact, f_l^r and f_l^a are the summation of the flow of regular and autonomous vehicles on all routes containing link l,

$$f_l^r = \sum_{p \in \mathcal{P}: l \in p} f_p^r, \text{ and } f_l^a = \sum_{p \in \mathcal{P}: l \in p} f_p^a.$$

Note that if all vehicles are either regular or autonomous for an O/D pair $w \in W$, i.e. either $\alpha_w = 0$ or $\alpha_w = 1$, then, we only have a single class of vehicles along that O/D pair, and for each path $p \in \mathcal{P}_w$, either $f_p = f_p^r$ or $f_p = f_p^a$. If for all network O/D pairs $w \in W$, the autonomy fraction $\alpha_w = 0$, then $f_l = f_l^r$ for all links $l \in L$. In fact, if all vehicles are regular, our routing game reduces to a single class game.

$$(\forall w \in W,\ \alpha_w = 0) \Longleftrightarrow \left(\forall p \in \mathcal{P},\ f_p = f_p^r\right). \tag{2}$$

In order to be able to model the incurred delays when vehicles are routed throughout the network, it is assumed that each link $l \in L$ has a delay per unit of flow function $e_l : \mathbb{R}^2 \to \mathbb{R}$. We assume that the delay per unit of flow for each path $p \in \mathcal{P}$ is obtained by the summation of the link delays over the links that form p,

$$e_p(f) = \sum_{l \in L: l \in p} e_l(f_l^r, f_l^a). \tag{3}$$

Equation (3) implies that the delay of each path $p \in \mathcal{P}$ depends not only on the flows of regular and autonomous vehicles along path p, but also on the flows along other paths. The overall network delay or social delay is given by

$$J(f) = \sum_{p \in \mathcal{P}} f_p e_p(f). \tag{4}$$

A. Wardrop Equilibrium

It is well known in the transportation literature that if there are many noncooperative agents, namely, flows that behave selfishly [24], a network is at an equilibrium if the well known Wardrop conditions hold [22]. Wardrop conditions state that at equilibrium, no user has any incentive for unilaterally changing their path. This implies that for an equilibrium flow vector f, if there exists a path $p \in \mathcal{P}_w$ such that either $f_p^r \neq 0$ or $f_p^a \neq 0$, we must have that $e_p(f) \leq e_{p'}(f)$ for all paths $p' \in \mathcal{P}_w$.

Definition 1. Given a network $G = (N, L, W)$, a feasible flow vector f is a Wardrop equilibrium if and only if for every O/D pair $w \in W$ and every pair of paths $p, p' \in \mathcal{P}_w$,

$$f_p^r \left(e_p(f) - e_{p'}(f)\right) \leq 0, \tag{5a}$$
$$f_p^a \left(e_p(f) - e_{p'}(f)\right) \leq 0. \tag{5b}$$

Note that an implication of the above definition is that for each O/D pair $w \in W$, and any two paths $p, p' \in \mathcal{P}_w$ such that $f_p \neq 0$ and $f_{p'} \neq 0$, we must have that $e_p(f) = e_{p'}(f)$.

Definition 2. Given an equilibrium flow vector f for a network $G = (N, L, W)$, we define the delay of travel for each O/D pair $w \in W$ to be

$$e_w(f) := \min_{p \in \mathcal{P}_w} e_p(f). \tag{6}$$

Motivated by the above discussion, $e_w(f)$ is precisely the delay across all paths $p \in \mathcal{P}_w$ which have a nonzero flow. Moreover, the equilibrium condition implies that for a path $p \in \mathcal{P}_w$ with zero flow, we have $e_p(f) \geq e_w(f)$.

It is worth mentioning that when there are no autonomous vehicles, i.e. for all O/D pairs $w \in W, \alpha_w = 0$, since $f_p^r = f_p$ for all $p \in \mathcal{P}$, Conditions (5) reduce to

$$f_p \left(e_p(f) - e_{p'}(f)\right) \leq 0, \quad \forall w \in W,\ \forall p, p' \in \mathcal{P}_w. \tag{7}$$

B. Delay Characterization

We first specify the structure of our delay model. If there is only a single class of regular vehicles in the network, the US Bureau of Public Roads (BPR) [25] suggests the following form of delay functions.

Assumption 1. When network links are shared by only regular vehicles, the link delay functions $e_l : \mathbb{R} \to \mathbb{R}$ are of the following form

$$e_l(f_l) = \left(a_l + \gamma_l \left(\frac{f_l}{C_l}\right)^{\beta_l}\right), \tag{8}$$

where C_l is the capacity of link l, and a_l, γ_l, and β_l are nonnegative link parameters.

In practice, a_l is normally the free–flow travel time on link l, γ_l is a constant link parameter, and β_l is a positive integer ranging from 1 to 4. However, in the remainder of the paper, we only require that a_l and γ_l be nonnegative link parameters, and β_l be a positive integer. In order to characterize the delay functions in networks with mixed autonomy, where we have two classes of vehicles, we first need to model the impact of autonomous vehicles on link capacities. In each network link $l \in L$, the link capacity C_l restricts the maximum possible flow of vehicles. It was shown in [16] that in networks with mixed autonomy, C_l depends on the autonomy fraction of link l defined as

$$\alpha_l := \frac{f_l^a}{f_l^a + f_l^r}. \tag{9}$$

We use $C_l(\alpha_l)$ to emphasize this dependence. Let m_l and M_l be the capacity of link l when all vehicles are regular and autonomous respectively. Since autonomous vehicles are capable of maintaining shorter headways, it is normally the case that $\frac{m_l}{M_l} \leq 1$. When the two classes of regular and autonomous vehicles are present in the network, using the results in [16], we have

$$C_l(\alpha_l) = \frac{m_l M_l}{\alpha_l m_l + (1 - \alpha_l) M_l}. \tag{10}$$

We adopt this capacity model throughout this paper. Since for each link $l \in L$, $\alpha_l = \frac{f_l^a}{f_l^a + f_l^r}$ and $f_l = f_l^a + f_l^r$, using (10), for networks with mixed autonomy, the delay function (8) can be modified as

$$e_l(f_l^r, f_l^a) = \left(a_l + \gamma_l \left(\frac{f_l^r + f_l^a}{\frac{m_l M_l (f_l^r + f_l^a)}{m_l f_l^a + M_l f_l^r}}\right)^{\beta_l}\right) \tag{11}$$
$$= \left(a_l + \gamma_l \left(\frac{f_l^a}{M_l} + \frac{f_l^r}{m_l}\right)^{\beta_l}\right). \tag{12}$$

Note that when only regular vehicles are present in the network, for each link $l \in L$ since $f_l = f_l^r$, the link delay function reverts to

$$e_l(f_l) = \left(a_l + \gamma_l \left(\frac{f_l^r}{m_l}\right)^{\beta_l}\right). \tag{13}$$

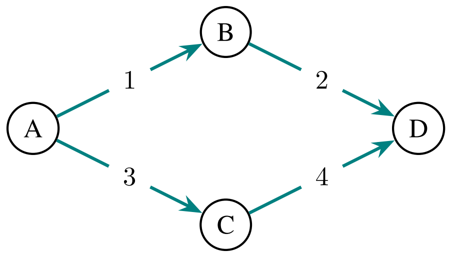

Fig. 1: A network with a single O/D pair and two paths.

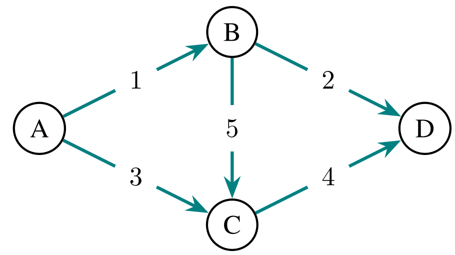

Fig. 2: A network with a single O/D pair (A to D) and three paths from A to D.

III. Prior Work

A. Existence of Equilibrium

We state the following proposition from the theorem in [26] which studies the conditions under which a Wardrop Equilibrium exists for a multiclass traffic network.

Proposition 1. *Given a network $G = (N, L, W)$, if the link delay functions are continuous and monotone in the link flow of each class; then, there exists at least one Wardrop equilibrium.*

Corollary 1. *Using (12), since our assumed delay functions are nonnegative, continuous, and monotone in the flow of each class, Proposition 1 implies that there always exists at least one Wardrop equilibrium for a routing game with mixed autonomy.*

B. Uniqueness of Equilibrium

In this part, we review the known results regarding the uniqueness of Wardrop Equilibria. When there is a single class of vehicles in the network, equilibrium uniqueness holds in a weak sense (See Theorem 3 from [19]).

Proposition 2. *Given a network G with a single class of vehicles for each O/D pair, if the delay functions are of the form (13), for a given demand vector r, we have*

1) *The equilibrium is unique in a weak sense, i.e. for each link l, the total flow f_l is unique for all Wardrop equilibrium flow vectors f.*
2) *For each O/D pair $w \in W$, the delay of travel $e_w(f)$ is unique for all Wardrop equilibrium flow vectors f. Thus, the delay of travel for each O/D pair at equilibrium, i.e. $e_w(f)$, only depends on network demand vector r. Hence, we may unambiguously define $e_w(r)$ to denote this unique value.*

C. Monotonicity of Social Delay

As we discussed above, in general, the equilibrium is not unique. However, if the conditions of Proposition 2 hold for a network, the social delay and the delay of travel for each O/D pair are unique. For a single class routing game on $G = (N, L, W)$, recall the following from Theorem 3 in [27].

Proposition 3. *Consider a network $G = (N, L, W)$, where only one class of vehicles exists for each O/D pair $w \in W$. Assume that for each link $l \in L$, $e_l(.)$ is continuous, positive valued, and monotonically increasing. Then, for each O/D pair $w \in W$, the delay of travel $e_w(r)$ is a continuous function of the demand vector r. Furthermore, $e_w(.)$ is nondecreasing in r_w when all other demands $r_{w'}, w' \neq w$, are fixed.*

IV. Uniqueness in Mixed-Autonomy Setting

Now we study equilibrium uniqueness in our setting. Using Corollary 1, we know that there exists at least one equilibrium. However, since in our setting, we have two classes of vehicles, Proposition 2 does not apply. Indeed, we demonstrate through an example that the equilibrium is *not* unique even in the weak sense introduced in Proposition 2.

Example 1. Consider the network of Figure 1. Let p_1 and p_2 be the ABD and ACD paths respectively. For each link $l = 1, \cdots, 4$, let the link parameters be $\beta_l = 1, a_l = 1, m_l = 1$, and $, M_l = 2$. Thus, for each link $l \in L$, the link delay function is $e_l(f_l^r, f_l^a) = 1 + f_l^r + \frac{f_l^a}{2}$. Assume that the demand from node A to D is $r = 2$, and $\alpha = 0.5$. The example is simple enough so that we can compute the equilibrium flows manually. Let f_1^r and f_1^a be the regular and autonomous flows along p_1, and f_2^r and f_2^a be the regular and autonomous flows along p_2. At equilibrium, using the symmetry of the network, we must have

$$\begin{aligned} 2 + 2f_1^r + f_1^a &= 2 + 2f_2^r + f_2^a \\ f_1^r + f_2^r &= 1 \\ f_1^a + f_2^a &= 1 \\ f_1^r, f_1^a, f_2^r, f_2^a &\geq 0. \end{aligned}$$

Clearly, there is no unique solution to the above set of equations. Moreover, among the set of all possible equilibrium flow vectors, for each link, the maximum link flow at equilibrium is 1.25, whereas the minimum link flow is 0.75 at equilibrium. This implies that equilibrium uniqueness does not hold in traffic networks with mixed autonomy.

V. Networks with a Single O/D Pair

In this section, we study two–terminal networks which have a single O/D pair. For such networks, since there is only one O/D pair, all paths originate from a common source o and end in a common destination d. Since W is singleton, we omit the subscript w from r_w, e_w and α_w throughout this section. Note that when the network has a single O/D pair, r and α are scalars.

Having observed that in the mixed-autonomy setting, the equilibrium is not unique, it is important to study if the social

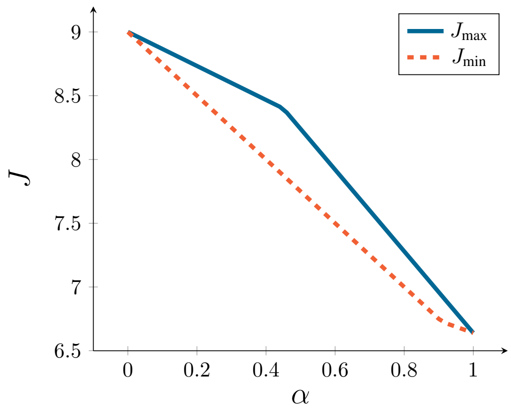

Fig. 3: Maximum and minimum social delay at equilibrium for Example 2.

delay at equilibrium is unique for all network equilibrium flow vectors. In the remainder of this paper, we use the term social delay to refer to the social delay of the network *at equilibrium*. In the following, we study the properties of the social delay including its uniqueness for networks with a single O/D pair. But, before proceeding, we need to define the notion of degree of road capacity asymmetry introduced in [16] via the following.

Definition 3. Given a network $G = (N, L, W)$, for each link $l \in L$, we define $\mu_l := m_l/M_l$ to be the degree of **road capacity asymmetry of link** l**.**

Note that since we assumed that autonomous vehicles' headway is less than or equal to that of regular vehicles, in the remainder of this paper, we assume that for each link $l \in L$, $\mu_l \le 1$. Using Definition 3, in the sequel, we consider two scenarios for investigating the properties of social delay:

1) Homogeneous degrees of road capacity asymmetry, where μ_l is the same for all links, i.e. $\mu_l = \mu$, for all links $l \in L$, where μ is the common value of capacity asymmetry.
2) Heterogeneous degrees of capacity asymmetry, where μ_l varies along different links.

A. Homogeneous Degrees of Capacity Asymmetry

In this case, we can establish the uniqueness of the social delay, and characterize the relationship between social delay and network autonomy ratio.

Theorem 1. *Given a network $G = (N, L, W)$ with a single O/D pair and a homogeneous degree of capacity asymmetry μ, for any value of demand $r > 0$, we have:*

1) For a fixed autonomy ratio $0 \le \alpha \le 1$, the social delay $J(f)$ is unique for all Wardrop equilibrium flow vectors f.
2) If for each $0 \le \alpha \le 1$, we denote the common value of social delay in the above by $J(\alpha)$, then $J(.)$ is continuous and nonincreasing.

Proof. Fix $r > 0$ and $0 \le \alpha \le 1$. Recalling Corollary 1, we know that a Wardrop equilibrium exists. Let $f = (f_p^r, f_p^a : p \in \mathcal{P})$ be such an equilibrium flow vector where $f_p = f_p^a + f_p^r$ for each path p in $\mathcal{P}$. Define $e_{\min}(f) := \min_{p\in\mathcal{P}} e_p(f)$. Since the network has only one O/D pair, and the delay associated with all paths with nonzero flows are the same, denoting this uniform path delay by $e_{\min}(f)$, we realize that the social delay is given by $J(f) = re_{\min}(f)$. For each path $p \in \mathcal{P}$, define the fictitious single-class regular flow $\tilde{f}_p := f_p^r + \mu f_p^a$. We claim that the flow vector $\tilde{f} = (\tilde{f}_p : p \in \mathcal{P})$ is a Wardrop equilibrium for a routing game on G with a single class of regular vehicles and a total demand of $\tilde{r} = r(1-\alpha) + r\alpha\mu$ with the delay function $(\tilde{e}_l : l \in L)$ defined as

$$\tilde{e}_l(\tilde{f}_l) = \left(a_l + \gamma_l \left(\frac{\tilde{f}_l}{m_l}\right)^{\beta_l}\right).$$

To see this, for each $p \in \mathcal{P}$, we show that relations (7) hold. Fix $p, p' \in \mathcal{P}$ and note that since f was a Wardrop equilibrium in the original setting, we know that $f_p^r(e_p(f) - e_{p'}(f)) \le 0$, and $f_p^a(e_p(f) - e_{p'}(f)) \le 0$. Multiplying the latter by the positive constant μ and adding the two inequalities, for every pair of paths $p, p' \in \mathcal{P}$, we have

$$\tilde{f}_p(e_p(f) - e_{p'}(f)) \le 0. \tag{14}$$

Now, we claim that for all $p \in \mathcal{P}$, we have $e_p(f) = \tilde{e}_p(\tilde{f})$. Note that for each link $l \in L$, we have $\tilde{f}_l = f_l^r + \mu f_l^a$. Using the fact that $\mu = m_l/M_l$ for all links $l \in L$, we get

$$\begin{aligned} \tilde{e}_p(\tilde{f}) &= \sum_{l\in p}\left(a_l + \gamma_l\left(\frac{f_l^r + \frac{m_l}{M_l}f_l^a}{m_l}\right)^{\beta_l}\right) \\ &= \sum_{l\in p}\left(a_l + \gamma_l\left(\frac{f_l^r}{m_l} + \frac{f_l^a}{M_l}\right)^{\beta_l}\right) = e_p(f). \end{aligned} \tag{15}$$

Substituting into (14), we realize that for each pair of paths $p, p' \in \mathcal{P}$, we have

$$\tilde{f}_p(\tilde{e}_p(\tilde{f}) - \tilde{e}_{p'}(\tilde{f})) \le 0, \tag{16}$$

which means that $\tilde{f}$ is an equilibrium flow vector. Clearly, the total demand of this new routing game is $\tilde{r} = \sum_{p\in\mathcal{P}} \tilde{f}_p = \sum_{p\in\mathcal{P}} f_p^r + \mu f_p^a = (1-\alpha)r + \mu\alpha r$. Moreover, define $\tilde{e}_{\min}(\tilde{f})$ to be the minimum of $\tilde{e}_p(\tilde{f})$ among $p \in \mathcal{P}$. Since w is the single O/D pair of the network, $\tilde{e}_{\min}(\tilde{f})$ is indeed equal to $\tilde{e}_w(\tilde{f})$, the travel delay of the single O/D pair of the network associated with $\tilde{f}$. Note that Proposition 2 implies that $\tilde{e}_{\min}(\tilde{f})$ is a function of $\tilde{r}$ only. On the other hand, (15) implies that $\tilde{e}_{\min}(\tilde{f}) = e_{\min}(f)$. Putting these together, we realize that

$$J(f) = re_{\min}(f) = r\tilde{e}_{\min}(\tilde{f}) = r\tilde{e}_w(\tilde{r}). \tag{17}$$

Note that the right hand side of the above identity does not depend on f, which establishes the proof of the first part of the theorem. In fact, this shows that

$$J(\alpha) = r\tilde{e}_w\left(r(1-\alpha) + \alpha\mu r\right).$$

From Proposition 3, we know that $\tilde{e}_w(.)$ is continuous and nondecreasing. Also, since $\mu \le 1$, the map $r \mapsto r(1-\alpha) + \alpha\mu r$ is continuous and nonincreasing. This completes the proof of the second part of the theorem. ∎

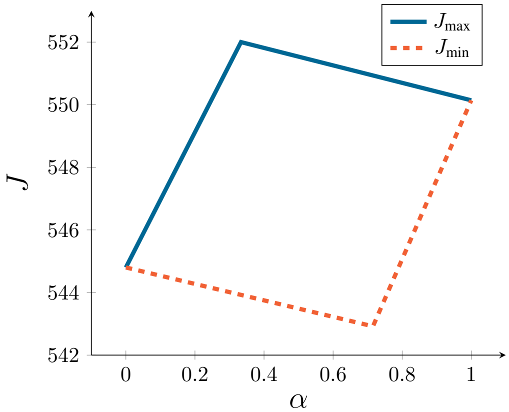

Fig. 4: Maximum and minimum social delays at equilibrium for Example 4.

B. Heterogeneous Degrees of Capacity Asymmetry

Now, we allow μ_l to vary among the network links. We show that this makes the behavior of the system more complex. First, we show via the following example that the social delay is not necessarily unique in this case.

Example 2. Consider the network shown in Figure 2. Assume that $\gamma_l = 1, \beta_l = 1$, for $l = 1, 2, \cdots, 5$. Let the other link parameters be the following: $\{a_1 = 1, m_1 = 1, M_1 = 1\}$, $\{a_2 = 2, m_2 = 1, M_2 = 3\}$, $\{a_3 = 1, m_3 = 1, M_3 = 2\}$, $\{a_4 = 1, m_4 = 1, M_4 = 4\}$, and $\{a_5 = 3, m_5 = 1, M_5 = 3\}$. Moreover, let the total flow from origin A to destination D be 2. Now, if we compute the social delay for this network for any $\alpha > 0$ at the different equilibria of the system, we observe that the social delay is *not* unique. In particular, Figure 3 shows the plots of the maximum and minimum social delay of the system at equilibrium for every value of α. To obtain Figure 3, we solved two optimization problems for finding maximum and minimum social delay subject to the equilibrium constraints using Mathematica. As Figure 3 shows, as soon as α starts to increase from 0, uniqueness of social delay is lost. Once, $\alpha = 1$, the uniqueness of social delay is again preserved.

Now, we study the effect of increasing the fraction of autonomous vehicles on the social delay. In the previous example, both the maximum and minimum social delays decreased as a function of α. But, is this necessarily the case? We use the following examples to demonstrate that it may not be true in general, as increasing network autonomy may increase social delay in some networks.

Example 3. Consider the network of Figure 2. Let $\gamma_l = 1$ and $\beta_l = 1$ for all links. Select the other network parameters to be the following, $\{a_1 = 0, m_1 = 0.1, M_1 = 0.1\}$, $\{a_2 = 50, m_2 = 1, M_2 = 1\}$, $\{a_3 = 50, m_3 = 1, M_3 = 1\}$, $\{a_4 = 0, m_4 = 0.1, M_4 = 0.1\}$, $\{a_5 = 10, m_5 = 0.5, M_5 = 1\}$. Let the total O/D demand be $r = 6$. In the absence of autonomy ($\alpha = 0$), the social delay is $J = 504.3$. However, if we increase the autonomy ratio to $\alpha = \frac{1}{10}$, $J = 518.6$. Clearly, in this case, the social delay increases when the network autonomy ratio α is increased. Note that since $\mu_l = 1$ for $l = 1, 2, 3, 4$ and $\mu_5 = 0.5 < 1$, this can be viewed as an instance of the classical Braess's Paradox [23], where an increase in the capacity of the middle link of a Wheatstone network can paradoxically lead to an increase in the social delay.

One might argue that if we allow μ_l to be strictly less than 1 for all network links $l \in L$, the network social delay will decrease by increasing the autonomy fraction. We use the following example to show that even in this case, increasing autonomy can worsen social delay.

Example 4. Consider the previous example with the total flow $r = 6$, but change M_l's to be, $M_1 = \frac{1}{9}$, $M_2 = 1.1$, $M_3 = 1.1$, $M_4 = \frac{1}{9}$, and $M_5 = 1$. In this case, clearly, $\mu_l < 1$, for all $l \in L$. We computed the maximum and minimum social delay at equilibrium for every autonomy fraction α. Figure 4 shows the maximum and minimum social delay in this example for different values of α. Figure 4 demonstrates that the maximum social delay increases as we increase α from 0, until we reach a local maximum. The minimum social delay decreases as we increase α from 0, until we reach a local minimum, and then, it increases sharply to values that are higher than the social delay at $\alpha = 0$. Surprisingly, when all vehicles are autonomous ($\alpha = 1$) the social delay is greater than the social delay when $\alpha = 0$, i.e. $J(\alpha = 1) > J(\alpha = 0)$. This might be counter intuitive as we expect the network with full autonomy to have smaller social delay. However, this example shows that when capacity increases are heterogeneous across the network, the selfish behavior of the vehicles when making their route choices might actually lead to worsening the social delay of the network.

As mentioned previously, the increase in social delay due to an increase in the fraction of autonomous vehicles is in fact similar to Braess's paradox. Braess's Paradox is the counterintuitive but well known fact that removing edges from a network or increasing the delay functions on certain links can improve the delay of all vehicles at equilibrium [24]. It was shown in previous studies that Braess paradox is prevalent in traffic networks [28] as the occurence of Braess's paradox heavily depends on network topology and the parameters of link delay functions [29], [30], [31]. However, the phenomenon that we observed in this paper for mixed–autonomy networks is different from the classical Braess's paradox in that link capacities are a function of the flows along links in a multiclass routing game. In other words, link capacity variations depend on how flows are routed throughout the network.

VI. Networks with Multiple O/D Pairs

So far, we have seen that even in a network with only one O/D pair, the introduction of autonomous vehicles can result in complex behaviors. Thus, it should be expected that a general network with multiple O/D pairs will exhibit similar counter intuitive behaviors. In the previous section, we saw that in a network with a single O/D pair, the existence of a homogeneous degree of capacity asymmetry throughout the network is sufficient for guaranteeing improvements in the social delay by increasing the fraction of autonomous vehicles.

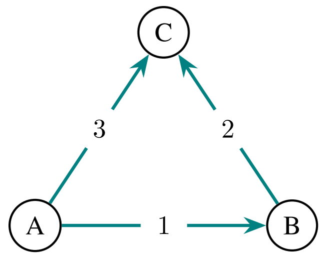

Fig. 5: A network with three O/D pairs.

We now show, via the following example, that this is not the case for networks with multiple O/D pairs.

Example 5. Consider the network shown in Figure 5 which was first introduced in [32]. There are three O/D pairs, $W = \{(\text{A,B}), (\text{B,C}), (\text{A,C})\}$. The total demand of the network O/D pairs are $r_{\text{AB}} = 17, r_{\text{AC}} = 20$, and $r_{\text{BC}} = 90$. Assume that $\gamma_l = 1$, $\beta_l = 1$, for all links $l \in L$. Let the other link parameters be $\{a_1 = 0, m_1 = 1, , M_1 = 4\}$, $\{a_2 = 0, m_2 = 1\}$, and $\{a_3 = 90, m_3 = 1\}$. Let the vehicles that travel from A to C, and from B to C be all regular vehicles, i.e. $\alpha_{\text{AC}} = \alpha_{\text{BC}} = 0$. Figure 6 shows a plot of the network social delay versus the fraction of autonomous vehicles traveling along O/D pair AB denoted by α_{AB}. As the figure shows, as the vehicle autonomy fraction along O/D pair AB increases, so does the social delay. Intuitively, by increasing the autonomy fraction along O/D pair AB, the travel delay along link 1 decreases. As a result, compared to the no-autonomy case, a higher number of the vehicles travelling form A to C are encouraged to take advantage of the lower delay along link 1. This in turn will lead to an increase in the travel delay along O/D pair BC. This increase combined with the high demand of vehicles along O/D pair BC leads to an increase in the overall social delay. This example shows that allowing vehicle autonomy along certain network O/D pairs can result in worsening the overall or social delay of the network even if the road degrees of capacity asymmetry are homogeneous. This is of paramount importance in practice. For instance, if O/D pair AB belongs to a high–income neighborhood, autonomous vehicles may first be deployed along this O/D pair, while other neighborhood or O/D pairs may still travel via regular vehicles. Then, although the early adoption of autonomous vehicles along O/D pair AB may lead to a decrease in travel delay of O/D pair AB, it worsens the social delay in the network and increases the delays experienced by users along other O/D pairs. This example shows that even with homogeneous degrees of capacity asymmetry, when there exist multiple O/D pairs, different autonomy fractions along network O/D pairs can be another source of heterogeneity in the network.

It was shown in [32], [33] that a decrease in the total demand of a single O/D pair, might lead to an increase in delay of travel along other network O/D pairs and the social delay. In the previous example, we showed that a similar behavior can also be observed due to the presence of autonomous vehicles. In fact, what we have shown so far is that the long known paradoxical traffic behavior resulting from constructing more roads or reducing demands can actually happen in networks

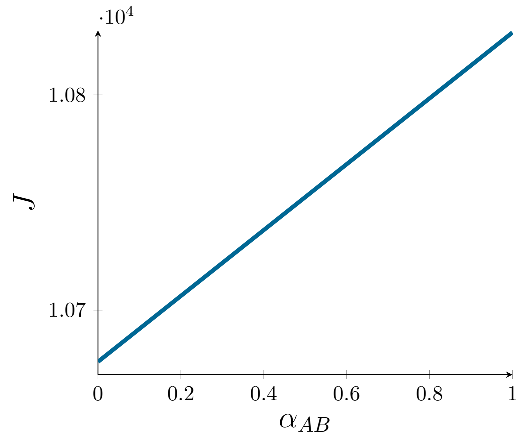

Fig. 6: Social delay in Example 5 for different fractions of autonomous vehicles traveling along O/D pair AB when vehicles along all other O/D pairs are regular.

with mixed autonomy due to the presence of autonomous vehicles. Thus, the mobility benefits of increasing vehicle autonomy in a network are not immediate.

VII. Bounding Performance Degradation

So far, we have shown that the social delay can increase as a consequence of the presence of autonomous vehicles in networks with mixed vehicle autonomy. Now, we wish to study whether we can bound this degradation in the network performance to estimate how much can social delay degrade by increasing the fraction of autonomous vehicles. To answer this, we derive a bound on the performance degradation that can result from all possible demand and autonomy fraction vectors in general networks that have a *homogeneous* degree of capacity asymmetry. To this end, for a given network G and a demand vector r, define the vector of fictitious reduced demand $\tilde{r} = (\tilde{r}_w : w \in W)$ to be $\tilde{r}_w = (1 - \alpha_w) r_w + \mu \alpha_w r_w$ for each O/D pair $w \in W$. Consider an auxiliary fictitious routing game with a total demand $\tilde{r}$ of only regular vehicles on G. For this auxiliary game, similar to Theorem 1, define $(\tilde{e}_l : l \in L)$ to be

$$\tilde{e}_l = \left(a_l + \gamma_l \left(\frac{\tilde{f}_l}{m_l} \right)^{\beta_l} \right), \tag{18}$$

and let $\tilde{e}_w(\tilde{r})$ be the delay of travel for each O/D pair $w \in W$ in this auxiliary game. Then, using the auxiliary fictitious game, we can state the following proposition.

Proposition 4. *Consider a general network $G = (N, L, W)$ with a homogeneous degree of capacity asymmetry $\mu \leq 1$ for of its links. For any demand vector r, fix the vector of autonomy fraction $\alpha = (\alpha_w : w \in W)$ such that $0 \leq \alpha_w \leq 1$ for all $w \in W$. Then, we have*

1) *The social delay $J(f)$ is unique for all Wardrop equilibrium flow vectors f.*
2) *The social delay of the original game is given by $J(f) = \sum_{w \in W} r_w \tilde{e}_w(\tilde{r}_w)$ for all Wardrop equilibrium flow vectors f.*

Proof. Fix r and α, such that for each O/D pair $w \in W$, $0 < r_w$ and $0 \leq \alpha_w \leq 1$. Recalling Corollary 1, we know that there exists at least one equilibrium. Let $f = (f_p^r, f_p^a : p \in \mathcal{P})$ be such an equilibrium flow vector for G. For each path $p \in \mathcal{P}$, define $\tilde{f}_p := f_p^r + \mu f_p^a$. By generalizing the proof of Theorem 1, it is easy to see that $\tilde{f} = (\tilde{f}_p : p \in \mathcal{P})$ is an equilibrium for the defined auxiliary routing game on G with reduced demand $\tilde{r}$ of only regular vehicles. Moreover, for each path $p \in \mathcal{P}$, $e_p(f) = \tilde{e}_p(\tilde{f})$. Therefore, for each O/D pair $w \in W$, $\tilde{e}_w(\tilde{f}) = \min_{p \in \mathcal{P}_w} \tilde{e}_p(\tilde{f}) = \min_{p \in \mathcal{P}_w} e_p(f) = e_w(f)$. Hence,

$$J(f) = \sum_{w \in W} r_w e_w(f) = \sum_{w \in W} r_w \tilde{e}_w(\tilde{f}). \tag{19}$$

Since $\tilde{f}$ contains only regular vehicles, recalling Proposition 2, for each $w \in W$, the delay of travel $\tilde{e}_w(\tilde{f})$ is unique for a given $\tilde{r}$. Thus, following a derivation similar to (17), we have

$$J(f) = \sum_{w \in W} r_w \tilde{e}_w(\tilde{r}). \tag{20}$$

As $\tilde{r}$ is uniquely determined for a given demand vector r and a vector of autonomy fraction α, the social delay $J(f)$ is unique for all Wardrop equilibrium flow vectors f and can be obtained via (20). $\square$

The uniqueness of social delay established by Proposition 4 implies that for a fixed demand vector r, the social delay is a well defined function of autonomy fraction α. With a slight abuse of notation, we use $J(\alpha)$ to emphasize the dependence of the social delay on the vector of autonomy fraction α. Note that Proposition 4 establishes a connection between our original routing game, which has two classes of vehicles, with a fictitious auxiliary routing game, which has only regular vehicles and a reduced demand vector $\tilde{r}$. We exploit this connection in the remainder of the paper. Since the auxiliary game has only one class of vehicles, the results in [34] hold for this game. Before proceeding, we need to adopt and review some of the definitions in [34] for the auxiliary game.

In the auxiliary game, for a given demand vector $\tilde{r}$, a flow vector $\tilde{f}$ is feasible if $\tilde{f}_p \geq 0$ for all paths $p \in \mathcal{P}$, and $\sum_{p \in \mathcal{P}_w} \tilde{f}_p = \tilde{r}_w$ for all O/D pairs $w \in W$. Let $\phi \in \mathbb{R}^{|L|}$ be a vector of *link* flows that result from a feasible flow vector $\tilde{f}$, where $|L|$ is the number of links in the network. Also, let Φ represent the set of all feasible link flow vectors ϕ for a given reduced demand vector $\tilde{r}$. Then, for a vector of link delay functions $(\tilde{e}_l : l \in L)$ of the form (18) and any vector $v \in \Phi$, define

$$\lambda\big((\tilde{e}_l : l \in L), v\big) := \max_{x \in \mathbb{R}^{|L|}_{\geq 0}} \frac{\sum_{l \in L} \big(\tilde{e}_l(v_l) - \tilde{e}_l(x_l)\big) x_l}{\sum_{l \in L} \tilde{e}_l(v_l) v_l}, \tag{21}$$

where $0/0$ is considered to be 0. Additionally, let $\tilde{\mathcal{E}}$ be the class of delay functions represented by (18). Define

$$\lambda(\tilde{\mathcal{E}}) := \sup_{(\tilde{e}_l : l \in L) \in \tilde{\mathcal{E}}, v \in \Phi} \lambda\left((\tilde{e}_l : l \in L), v\right). \tag{22}$$

It is important to mention that since the class of delay functions $\tilde{\mathcal{E}}$ is monotone, $\lambda(\tilde{\mathcal{E}}) \leq 1$ (See Section 4 in [34]). Note that $\lambda(\tilde{\mathcal{E}})$ can be easily computed for certain classes of delay functions such as polynomials. For instance, $\lambda(\tilde{\mathcal{E}}) = \frac{1}{4}$ for the class of linear delay functions.

Now, we can bound the network performance degradation due to the introduction of autonomy in homogeneous networks via the following theorem.

Theorem 2. *Consider a general network $G = (N, L, W)$ with a homogeneous degree of capacity asymmetry μ. Fix the demand vector r. Let J^o be the social delay when all vehicles are nonautonomous, i.e. $\alpha_w = 0$ for all O/D pairs $w \in W$. Then, for any vector of autonomy fraction α such that $0 \leq \alpha_w \leq 1$ for all $w \in W$, we have*

$$J(\alpha) \leq (1 - \lambda(\tilde{\mathcal{E}}))^{-1} J^o, \tag{23}$$

where $J(\alpha)$ is the social delay for a given vector of autonomy fraction α.

Proof. Fix the demand vector r. Let $f^o = (f_p^o : p \in \mathcal{P})$ be an equilibrium flow vector when all vehicles are regular. We further use f_l^o to denote the flow along link $l \in L$ in this case. Note that using Proposition 2 , we know that at equilibrium, f_l^o is unique for every link $l \in L$. Moreover, for each path $p \in P$, we use e_p^o to represent the delay along path p when all vehicles are regular. Using Proposition 2, in the absence of autonomy, the delay of travel for each O/D pair $w \in W$ is unique. Thus, in the no-autonomy case, the unique social delay is $J^o = \sum_{w \in W} r_w e_w^o(r)$, where $e_w^o(r)$ is the delay of travel for O/D pair $w \in W$ when all vehicles are regular.

On the other hand, when there are autonomous vehicles with a given autonomy fraction α in the network, as defined in Proposition 4, construct the auxiliary game on G with fictitious reduced demand $\tilde{r} = (\tilde{r}_w : w \in W)$ of only regular vehicles, where $\tilde{r}_w = (1 - \alpha_w) r_w + \mu r_w \alpha_w$ for every $w \in W$. Let $\tilde{f} = (\tilde{f}_p : p \in \mathcal{P})$ be an equilibrium flow vector for this auxiliary game. Using Proposition 4, the social delay of the network with autonomy is given by $J(\alpha) = \sum_{w \in W} r_w \tilde{e}_w(\tilde{r})$. First, we claim that

$$J(\alpha) = \sum_{w \in W} r_w \tilde{e}_w(\tilde{r}) \leq \sum_{l \in L} f_l^o \tilde{e}_l(\tilde{r}). \tag{24}$$

To see this, note that for every link $l \in L$, we have $f_l^o = \sum_{p \in \mathcal{P} : l \in p} f_p^o$. Furthermore, the origin and destination of each path $p \in \mathcal{P}$ are unique. Hence, each path p belongs to one and exactly one O/D pair $w \in W$. Consequently, $f_l^o = \sum_{w \in W} \sum_{p \in \mathcal{P}_w : l \in p} f_p^o$, and we have

$$\begin{aligned}
\sum_{l \in L} f_l^o \tilde{e}_l(\tilde{r}) &= \sum_{l \in L} \left(\sum_{w \in W} \sum_{p \in \mathcal{P}_w : l \in p} f_p^o \right) \tilde{e}_l(\tilde{r}) \\
&= \sum_{w \in W} \sum_{l \in L} \left(\sum_{p \in \mathcal{P}_w : l \in p} f_p^o \right) \tilde{e}_l(\tilde{r}) \\
&= \sum_{w \in W} \sum_{p \in \mathcal{P}_w} f_p^o \sum_{l : l \in p} \tilde{e}_l(\tilde{r}) \\
&= \sum_{w \in W} \sum_{p \in \mathcal{P}_w} f_p^o \tilde{e}_p(\tilde{r}),
\end{aligned}$$

where $\tilde{e}_p(\tilde{r})$ is the delay of travel along path $p \in \mathcal{P}_w$ in the auxiliary game. Recalling Definition 2, for the auxiliary game, the travel delay of an O/D pair $w \in W$ is given by $\tilde{e}_w(\tilde{r}) = \min_{p\in\mathcal{P}_w} \tilde{e}_p(\tilde{r})$; thus, we have

$$\begin{aligned}\sum_{w\in W}\sum_{p\in\mathcal{P}_w} f_p^o \tilde{e}_p(\tilde{r}) &\geq \sum_{w\in W}\sum_{p\in\mathcal{P}_w} f_p^o \tilde{e}_w(\tilde{r}) \\ &= \sum_{w\in W} \tilde{e}_w(\tilde{r}) \sum_{p\in\mathcal{P}_w} f_p^o \\ &= \sum_{w\in W} r_w \tilde{e}_w(\tilde{r}),\end{aligned}$$

which proves our claim in (24). Now, since the auxiliary game has only one class of vehicles, we can use Lemma 4.1 from [34]. More precisely, since $\tilde{f}$ is an equilibrium for the auxiliary game, then Lemma 4.1 from [34] states that for every nonnegative vector of link flows $x \in \mathbb{R}^{|L|}_{\geq 0}$ (x is not necessarily a feasible link flow vector), we have

$$\sum_{l\in L} x_l \tilde{e}_l(\tilde{f}_l) \leq \sum_{l\in L} x_l \tilde{e}_l(x_l) + \lambda(\tilde{\mathcal{E}}) \sum_{l\in L} \tilde{f}_l \tilde{e}_l(\tilde{f}_l). \tag{25}$$

Since f_l^o is nonnegative for every link $l \in L$, substituting x_l by f_l^o in (25), we get

$$\sum_{l\in L} f_l^o \tilde{e}_l(\tilde{f}_l) \leq \sum_{l\in L} f_l^o \tilde{e}_l(f_l^o) + \lambda(\tilde{\mathcal{E}}) \sum_{l\in L} \tilde{f}_l \tilde{e}_l(\tilde{f}_l). \tag{26}$$

Now, note that since both the auxiliary game and the game with no autonomy have only regular vehicles, utilizing (18), we realize that

$$\begin{aligned}\tilde{e}_l(f_l^o) &= \left(a_l + \gamma_l \left(\frac{f_l^o}{m_l}\right)^{\beta_l}\right) \\ &= e_l^o(f_l^o).\end{aligned}$$

Thus,

$$\sum_{l\in L} f_l^o \tilde{e}_l(f_l^o) = \sum_{l\in L} f_l^o e_l^o(f_l^o) = J^o. \tag{27}$$

Now, since $J(\alpha) = \sum_{w\in W} r_w \tilde{e}_w(\tilde{r})$, using (24), (26), and (27), we realize that

$$J(\alpha) \leq J^o + \lambda(\tilde{\mathcal{E}}) \sum_{l\in L} \tilde{f}_l \tilde{e}_l(\tilde{r}). \tag{28}$$

As $\tilde{f}$ is an equilibrium for the auxiliary routing game, $\sum_{l\in L} \tilde{f}_l \tilde{e}_l(\tilde{r}) = \sum_{w\in W} \tilde{r}_w \tilde{e}_w(\tilde{r})$. Since for each O/D pair $w \in W$, we assumed that $\alpha_w \leq 1$, we can conclude that for each O/D pair $w \in W$, we have $\tilde{r}_w \leq r_w$. Therefore, using Proposition 4, we realize that

$$\sum_{w\in W} \tilde{r}_w \tilde{e}_w(\tilde{r}) \leq \sum_{w\in W} r_w \tilde{e}_w(\tilde{r}) = J(\alpha). \tag{29}$$

Using (29) and (28), we get

$$J(\alpha) \leq J^o + \lambda(\tilde{\mathcal{E}}) J(\alpha). \tag{30}$$

Hence, for the our monotone class of delay functions $\tilde{\mathcal{E}}$ with $\lambda(\tilde{\mathcal{E}}) < 1$, we can conclude that

$$J(\alpha) \leq (1 - \lambda(\tilde{\mathcal{E}}))^{-1} J^o,$$

which completes the proof. □

Theorem 2 provides an upper bound on the severity of increases in traffic delays when a fraction of regular vehicles is replaced by autonomous vehicles.

We now postulate, as an analogous concept to the price of anarchy [35], the price of vehicle autonomy in homogeneous networks under every demand vector r as follows:

$$\eta := \max_{\alpha:\ \forall w,\ 0\leq\alpha_w\leq 1} \frac{J(\alpha)}{J^o}. \tag{31}$$

Theorem 2 indicates that $\eta \leq (1 - \lambda(\tilde{\mathcal{E}}))^{-1}$. For polynomial delay functions of degree less than or equal to 4, $(1-\lambda(\tilde{\mathcal{E}}))^{-1} = 2.151$ [34]. The bound that we have derived for the price of vehicle autonomy is similar to the bounds derived for the price of anarchy of routing games with a single class of users in [35]. However, unlike the bounds for price of anarchy, the tightness of our bound for η needs further investigation.

VIII. Conclusion and Future Work

In this paper, we studied how the coexistence of autonomous and regular vehicles in traffic networks will affect network mobility when all vehicles select their routes selfishly. We compared the total network delay at a Wardrop equilibrium in networks with mixed autonomy with that of the networks with only regular vehicles. Having shown that the equilibrium is not unique in the mixed–autonomy setting, we proved that the total delay is unique when the road degree of capacity asymmetry, which is the ratio between the roadway capacity with only regular vehicles and the roadway capacity with only autonomous vehicles, is homogeneous among its roadway. We further proved that the total delay is a nonincreasing and continuous function of the fraction of autonomous vehicles on the roadways (aka the autonomy ratio) when the network has only one O/D pair. However, we showed that allowing for heterogeneous degrees of capacity asymmetry or multiple O/D pairs in the network results in counter intuitive behaviors such as the fact that increasing network autonomy ratio can worsen the network total delay. Finally, we derived an upper bound for the "price of vehicles autonomy" in networks with a homogeneous degree of capacity asymmetry, which estimates the worst possible increase in network social delay due to the presence of autonomous vehicles.

We believe that the results presented in this paper indicate that the expected mobility benefits resulting from the wide spread utilization of autonomous vehicles in traffic networks may not be immediate. Thus, in order to take advantage of the potential mobility benefits of autonomy, it will be necessary to study traffic management and control strategies for mixed–autonomy networks. Further control and management must be developed for these networks such that the system is steered to the equilibria that have lower total delay. Therefore, revisiting routing and tolling strategies for networks with mixed vehicle autonomy is essential.

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