A Submodular Approach for Optimal Sensor Placement in Traffic Networks

Negar Mehr[1] and Roberto Horowitz[2].

***Abstract*— Precious measurements from infrastructure traffic sensors versus their high–priced installation and maintenance costs, make sensor placement a paramount problem for traffic networks. We study sensor placement for traffic networks in two settings. When no routing information is available, we propose to maximize the identifiability of all link flows in the network. When the network routing is known, we use a previously defined metric of minimizing estimation error of a BLUE (Best Linear Unbiased Estimator) estimator. We prove that in both cases, the problem is submodular. By exploiting the submodularity of the problem, we can use polynomial–time approximations of the combinatorial placement problem with guaranteed optimality bounds, which is of importance due to the inherent large–scale nature of transportation networks. We demonstrate the performance of our method in a grid example network.**

I. INTRODUCTION

A key challenge in designing traffic networks is to decide where the infrastructure sensors such as loop detectors must be located so as to provide a set of beneficial measurements for the system. Normally, deploying a larger set of sensors lead to a superior performance. However, installation and maintenance costs of sensors are not negligible. Thus, a trade–off between system performance and the incurred cost must be sought when designing networks.

The sensor placement problem has received significant attention in different communities. In the transportation context, efficiency of the majority of the proposed control algorithms for improving traffic conditions, such as ramp metering and signal control, depends heavily on the existence of accurate estimates of system states [1], [2]. Among different types of sensors, loop detectors are the primary source of information and measurements used in estimating traffic state and flows. Due to the high expenses of installing and maintaining these sensors; however, the network sensors are desired to be as sparse as possible [3].

Sensor placement has been considered in the traffic literature from different view points. A large body of the literature focuses on sensor placement for providing the highest accuracy when estimating origin destination flows [4], [5], [6]. To this end, the problem is formulated as a mixed–integer linear program. In [7], the expected information theoretic gain for identifying origin–destination flows of the network is maximized. There are other works including [8], [9] that focus on capturing the probabilistic models of sensor failures in the placement design. Placement of hubs has also been considered in [10] for efficacy of transportation.

The aforementioned approaches have been concentrating on how to formulate the problem such that certain aspects of sensor placement are addressed. Nevertheless, when it comes to solving the proposed optimization problems, the majority of them lead to linear or nonlinear mixed–integer problems that are hard or impossible to solve for general networks of arbitrary size. This is not a surprise as the problem is combinatorial at heart. There have been few works focusing on how to actually solve the optimization problems efficiently. In [3], the combinatorial optimization problem is converted to a continuous convex program. Nonetheless, for obtaining reasonable results, several parameters need to be hand tuned.

In this paper, we discuss how the sensor placement problem for traffic networks can be easily solved via submodular optimization. Submodular optimization is a strong tool in subset selection problems that allow for polynomial–time approximating algorithms with guarantees on optimality bounds. Submodular optimization has been widely used in sensor and actuator placement of linear and nonlinear dynamical systems [11], [12], [13], [14], [15], [16]. In these works, the system dynamics are either linear or linearized, and the maximization of either controllability or observability of the system is considered. Due to the complex hybrid dynamics of traffic networks, such approaches do not directly apply. As a result, several alternative formulations that do not directly take dynamics into account are proposed. In this paper, we show that these alternative formulations exhibit very similar submodular behavior to that of controllability/observability measures of linear dynamical systems.

We consider the problem in two cases: no routing information is available while having perfect sensors versus the case where turning ratios are available, and sensor noises are taken into account. We describe the problem in these two cases and prove that the sensor placement problem has a submodular behavior; therefore, greedy algorithms can be utilized for efficiently solving these problems. This is a practical result as it allows for scaling up the problem to networks of arbitrary size and provides a systematic and formal way of solving the problem with no need for tuning or trial and errors. We demonstrate the practicality and ability of our framework for an example network.

The organization of this paper is as follows. In Section II, we describe our notation and the required background. Network model is described in Section III. Next, we describe the problem formulation in Section IV. In Section V, we prove the submodular nature of the problem. In Section VI,

[1]Negar Mehr is with the Mechanical Engineering Department, University of California, Berkeley, 410 McLaughlin Hall, Berkeley, CA, USA negar.mehr@berkeley.edu

[2]Roberto Horowitz is with the Mechanical Engineering Department, University of California, Berkeley, 6143 Etcheverry Hall, Berkeley, CA, USA horowitz@berkeley.edu

we demonstrate examples of how our framework performs on a sample network, and we finally conclude the paper in Section VII.

II. Notation and Background

A. Notation

$\mathbb{R}$ is the set of real numbers. $\mathbb{R}_{\succ 0}$ is the space of symmetric positive definite matrices of relevant dimension. To distinguish matrices from vectors, upper case letters are exclusively used for matrices. For a vector x, x_i is the i_{th} element of the vector. For a matrix M, M^T is the matrix transpose, and $\text{Im}(M)$ is image of M. M^+ is pseudoinverse of M. Moreover, M_{ij} is the element at the i_{th} row and j_{th} column of matrix M. The identity matrix is denoted by I. For a finite set $\mathcal{C}$, $|\mathcal{C}|$ is its cardinality. Finally, for two sets $\mathcal{C}_1$ and $\mathcal{C}_2$, $\mathcal{C}_1 \backslash \mathcal{C}_2$ represents set difference.

B. Background on Submodularity

Consider a finite set $\mathcal{V}$. The set of all subsets of $\mathcal{V}$ is denoted by $2^{\mathcal{V}}$. Consider the set functions of the form $g : 2^{\mathcal{V}} \to \mathbb{R}$. For such functions, monotonicity is defined as:

Definition 1. *A set function g is monotone decreasing if for all $\mathcal{V}_1, \mathcal{V}_2 \subseteq \mathcal{V}$, the following is true:*

$$\mathcal{V}_1 \subseteq \mathcal{V}_2 \quad \textit{if and only if} \quad g(\mathcal{V}_1) \geq g(\mathcal{V}_2). \tag{1}$$

A similar definition of monotonicity holds for matrix functions.

Definition 2. *For $\tilde{g} : \mathbb{R}_{\succ 0} \to \mathbb{R}$, $\tilde{g}$ is monotone decreasing if $Q_1 \preceq Q_2 \Longleftrightarrow \tilde{g}(Q_1) \geq \tilde{g}(Q_2)$.*

Definition 3 (cf. [17])**.** *A set function $g : 2^{\mathcal{V}} \to \mathbb{R}$ is submodular if, for any two sets $\mathcal{V}_1$ and $\mathcal{V}_2$ such that $\mathcal{V}_1 \subseteq \mathcal{V}_2 \subseteq \mathcal{V}$, and any $v \notin \mathcal{V}_2$, we have:*

$$g(\mathcal{V}_1 \cup v) - g(\mathcal{V}_1) \geq g(\mathcal{V}_2 \cup v) - g(\mathcal{V}_2). \tag{2}$$

Equation (2) is interpreted as having a diminishing returns property [18] which implies that addition of an element v to a smaller set, which is a subset of the larger set, will lead to a larger increase in g. To enhance readability, we adopt the notation introduced in [19] for the obtained gain when an element v is added to a set $\mathcal{S}$: $\Delta(v \,|\, \mathcal{S}) = g(\mathcal{S} \cup v) - g(\mathcal{S})$.

The following theorem from [20] is a practically powerful result in proving submodularity of set functions.

Theorem 1. *A set function $g : 2^{\mathcal{V}} \to \mathbb{R}$ is submodular if and only if the set functions $g_v : 2^{\mathcal{S}-v} \to \mathbb{R}$ defined as $g_v(\mathcal{S}) = g(\mathcal{S} \cup v) - g(\mathcal{S})$ are monotone decreasing $\forall v \in \mathcal{V}$.*

C. Submodular Optimization

When it comes to solving the optimization problems of the form:

$$\begin{aligned} \underset{\mathcal{S} \subseteq \mathcal{V}}{\text{maximize}} \quad & g(\mathcal{S}) \\ \text{subject to} \quad & |\mathcal{S}| \leq k, \end{aligned} \tag{3}$$

where k is the maximum allowed cardinality, polynomial time algorithms can be utilized with certain optimality bounds provided that the function g has a submodular structure [18]. In particular, when g is submodular, Algorithm 1 can be used for solving (3). What Algorithm 1 does is that at each iteration, it simply adds the element which maximizes the gain $\Delta(v|\mathcal{S})$. It terminates when it iterates for k times or encounters a scenario when adding an element will not lead to a performance improvement.

Algorithm 1 Greedy algorithm for cardinality–constrained optimization problems.

```
procedure GREEDYMAX(k, g(.))
    Initialize:
    S ← ∅, i ← 0
    while i ≤ k do
        v_i ← argmax_{v∈(V\S)} (Δ(v|S))
        if Δ(v_i|S) ≤ 0 then
            return S
        else
            S_i ← S_i ∪ v_i
            i ← i + 1
        end if
    end while
    return S
end procedure
```

The optimality bound of Algorithm 1 is characterized in [17] via the following theorem.

Theorem 2. *Let the optimal solution to* (3) *be $\mathcal{S}^*$ and S be the set obtained from running the greedy algorithm. Then, if the function g is a monotone set function, letting e to denote the Napier's constant, the following is always true:*

$$g(\mathcal{S}) \geq (1 - \frac{1}{e}) g(\mathcal{S}^*). \tag{4}$$

It is important to mention that the above optimality bound is a conservative one as the greedy algorithm generally performs much better in practice.

III. Network Model

In this paper, we model traffic networks similarly to Ponit–Q models [21], but the modeling framework is general enough to include freeways as well. We assume that the network can be represented by a graph $\mathcal{G} = (\mathcal{N}, \mathcal{L})$, where $\mathcal{N}$ is the set of network nodes, and $\mathcal{L}$ is the set of network links. Let $N = |\mathcal{N}|$ and $L = |\mathcal{L}|$ be the number of network nodes and links respectively. For each node $n \in \mathcal{N}$, $\mathcal{I}(n)$ is the set of links input to the node n; while $\mathcal{O}(n)$ is the set of links leaving n. Network links are categorized into three different types: entry links $\mathcal{L}_{entry}$, internal links $\mathcal{L}_{inter}$ and exit links $\mathcal{L}_{exit}$. Entry links do not have a starting node. Exogenous demands enter the network through entry links. Internal links connect network nodes, and exit links are the ones through which vehicles leave the network. Exit links do not have any end node. We denote the number of entry, internal and exit links respectively by $|\mathcal{L}_{\text{entry}}| = L_{\text{entry}}$, $|\mathcal{L}_{\text{inter}}| = L_{\text{inter}}$, and $|\mathcal{L}_{\text{exit}}| = L_{\text{exit}}$.

As we model the network from a macroscopic point of view, for each link l, we let f_l be the amount of vehicular flow on link l (vehicles per unit of time). The exogenous vehicular demand on each entry link l is denoted by d_l. At each junction n, certain movements are allowed. For such movements, let $f(l,m)$ be the flow of vehicles moving from link l to m. Moreover, the fraction of vehicles leaving link l to link m is denoted by $r(l,m)$. The set of $r(l,m)$'s for all $l,m \in \mathcal{L}$, known as network turn ratios, describe how vehicles are routed throughout the network.

Since no vehicle is normally stored in traffic junctions, conservation of flow must hold at the network junctions.

$$\sum_{l\in\mathcal{I}(n)} f_l = \sum_{m\in\mathcal{O}(n)} f_m, \forall n \in \mathcal{N}. \tag{5}$$

In order for Equation (5) to hold, we must have the following:

$$\begin{align} f_l &= d_l, \quad l \in \mathcal{L}_{entry}, \tag{6a}\\ f(l,m) &= r(l,m) f_l, \tag{6b}\\ f_l &= \sum_m f(l,m), \quad l \in \mathcal{L}_{entry} \cup \mathcal{L}_{inter}, \tag{6c}\\ f_m &= \sum_l f(l,m), \quad m \in \mathcal{L}_{inter} \cup \mathcal{L}_{exit}. \tag{6d}\end{align}$$

To have a compact notation, we introduce a vectorized format of our notation. Let $f \in \mathbb{R}^L$ be the vector of link flows for all links in the network. Likewise, we use $d \in \mathbb{R}^L$ to represent the vector of exogenous network demands such that d_l is equal to the exogenous arrival on link l if $l \in \mathcal{L}_{entry}$ and $d_l = 0$ otherwise. Additionally, we can collect all turn ratios in a matrix $R \in \mathbb{R}^{L\times L}$ such that $R_{lm} = r(l,m),\ \forall l,m \in \mathcal{L}$. Using this notation, the flow Equations (6) can be compactly written as:

$$f = (I - R^T)^{-1} d. \tag{7}$$

We will use Equation (7) later as the required constraints that the estimated flows from sensor measurements must satisfy.

IV. Problem Formulation

Sensor placement deals with the problem of finding the best possible locations for sensors such that a trade–off between performance and cost is achieved. Such problems are normally formulated as budget constrained problems, where a limited number of sensors are available, and one needs to decide on the locations so as to optimize a performance metric. Assume that at most k sensors are available. Among the set of network links $\mathcal{L}$, a certain subset $\mathcal{L}_m \subseteq \mathcal{L}, |\mathcal{L}_m| \leq k$, can be selected as the desired locations for placing the sensors. Assuming that there exits a function $u : 2^{\mathcal{L}} \to \mathbb{R}$ that maps possible sensor locations to their corresponding performance measures, the sensor placement problem can be formulated as follows:

$$\begin{aligned} \underset{\mathcal{L}_m\subseteq\mathcal{L}}{\text{maximize}} \quad & u(\mathcal{L}_m)\\ \text{subject to} \quad & |\mathcal{L}_m| \leq k. \end{aligned} \tag{8}$$

In the general setting of the traffic networks, an optimal sensor location should be the one that leads to the minimum density estimation error. However, due to the nonlinear dynamics of traffic networks [22], obtaining the function u that characterizes the density estimation error is non trivial since density estimation itself is achieved via filtering algorithms such as particle filtering [23]. Consequently, in this paper, precisely estimating link flows is considered as a proxy for estimating densities as in [3], [24].

V. Submodular Sensor Placement

As mentioned previously, we consider two scenarios: When the turning ratios matrix R is unknown, and when the turning ratios are available.

A. Case I: Unknown Turning Ratios

In this case, the goal is to locate sensors such that the maximum number of link flows can be identified from a set of measurements. In such a scenario, the goal is to maximize the identifiability of link flows provided that a set of k flow measurements are obtained. Consider the network graph $\mathcal{G} = (\mathcal{N}, \mathcal{L})$. Assume that the set $\mathcal{L}_m$ with $|\mathcal{L}_m| \leq k$ designates the candidate locations of sensors. Then, we have:

$$f_l = y_l, \quad \forall l \in \mathcal{L}_m, \tag{9}$$

where y_l is the value measured by the sensor located in l. Furthermore, we know that at each node, flow conservation (5) must hold. This implies that we have N linear equations that link flows must satisfy. In other words, we have the following system of linear equations:

$$\begin{align} f_l = y_l, \quad l \in \mathcal{L}_m, \tag{10a}\\ \sum_{l\in\mathcal{I}(n)} f_l = \sum_{m\in\mathcal{O}(n)} f_m, \quad \forall n \in \mathcal{N}. \tag{10b}\end{align}$$

This leads to a set of $N + |\mathcal{L}_m|$ linear equations, while the number of variables is L.

In [24], a geometric interpretation of this problem was proposed. Assuming that each link that is equipped with a flow measurement is removed from the graph, it was proved that in order to be able to identify the flows at all network links, the sensor locations must be chosen such that the remaining graph does not have any cycles. Thus, the minimum number of required sensors is achieved when the remaining network graph is a tree. However, removing all cycles still requires a large number of sensors which might not be feasible in practice. Therefore, when facing budget constraints, the goal would be to maximize the number of link flows that could actually be inferred from the measurements. To formalize this, concatenate the set of equations (10) as:

$$Af = 0, \tag{11}$$

where $A_{(N+|\mathcal{L}_m|)\times L}$ is the matrix of linear equations imposed on the vector of link flows f. Note that the unidentifiable flows are the ones that lie in the null space of matrix A. Hence, maximizing the number of identifiable flows can be encoded as shrinking the null space of A, or, equivalently, maximizing $\text{rank}(A)$. Thus, our placement problem can be formulated as:

$$\begin{aligned} \underset{\mathcal{L}_m \subseteq \mathcal{L}}{\text{maximize}} \quad & \text{rank}(A) \\ \text{subject to} \quad & |\mathcal{L}_m| \leq k. \end{aligned} \tag{12}$$

The solution to (12) can be easily found using the following theorem.

Theorem 3. *Consider $\mathcal{V} = \{a_1, a_2, \cdots, a_L\}$ to be the set of possible measurement rows of the matrix A attained from Equation (10.a). For a subset $\mathcal{S} \subseteq \mathcal{V}$, let $A_{\mathcal{S}}$ be the A matrix obtained by considering the measurement rows that correspond to the set $\mathcal{S}$. Then, the set function $u : 2^{\mathcal{V}} \to \mathbb{R}$ with $u(\mathcal{S})$ being $\text{rank}(A_{\mathcal{S}})$ is submodular.*

Proof. The main idea in this proof is that $\text{rank}(A_{\mathcal{S}})$ can only increase when $\mathcal{S}$ gets larger. For a given set $\mathcal{S}$ and a row a_i, $1 \leq i \leq L$, construct the gain functions $u_{a_i} : 2^{\mathcal{S} - a_i} \to \mathbb{R}$ as introduced earlier. We will then have:

$$u_{a_i}(\mathcal{S}) = \text{rank}(A_{\mathcal{S}} \cup a_i) - \text{rank}(A_{\mathcal{S}}) \tag{13}$$
$$= \text{rank}(a_i) - \dim(\text{Im}(A_{\mathcal{S}}) \cap \text{Im}(a_i)). \tag{14}$$

As $\text{rank}(a_i)$ is 1, and $\text{Im}(A_{\mathcal{S}})$ can only increase when $\mathcal{S}$ gets larger, we can conclude that $u_a(\mathcal{S})$ is monotone decreasing; hence, using Theorem 1, u is submodular. ☐

It is important to point the similarity of the above proof to that of maximizing the rank of controllability/observability matrix of linear dynamical systems in [12]. Also, note that in this formulation, sensor qualities are not taken into account, and only *identifiability* is considered.

B. Case II: Available Turning Ratios

The sensor placement problem has also been studied when some information about the network turning ratios is accessible through either surveys or counts from radar sensors [3]. In this setting, we employ the framework used in [3]. We know that link flows obey (7). Equation (7) is equivalent to $d = (I - R^T)f$. As mentioned previously, d_l is simply the demand on $\mathcal{L}_{\text{entry}}$ and zero elsewhere. As a result, if $P = (I - R^T)$, and we let $\bar{P}_{(L - L_{\text{entry}}) \times L}$ to be the matrix obtained by removing the rows that correspond to entry links in P, we should have:

$$\bar{P}f = 0. \tag{15}$$

Equation (15) provides the set of linear constraints that any candidate set of link flows must satisfy. Using our previous notation, let $\mathcal{L}_m, |\mathcal{L}_m| \leq k$ be the set of links selected for placing the sensors. Assume the following linear measurement model:

$$y = H_{\mathcal{L}_m} f + \omega, \tag{16}$$

where $y \in \mathbb{R}^{|\mathcal{L}_m|}$ is the vector of measurements provided by the mounted sensors. $H_{|\mathcal{L}_m| \times L}$ is a matrix of zero and ones such that $H_{\mathcal{L}_m} \mathbf{1}_L = \mathbf{1}_{|\mathcal{L}_m|}$ and $\mathbf{1}^T_{|\mathcal{L}_m|} H_{\mathcal{L}_m} \mathbf{1}_L = |\mathcal{L}_m|$ with i_{th} row indicating where the i_{th} sensor is located (each row is an array of zeros except for the link index where a sensor is located, which is set to 1). ω is the vector of random zero mean noise associated with the sensors set.

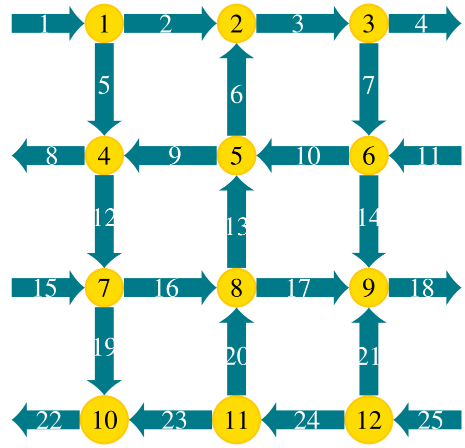

Fig. 1. The topology of the example network.

Consider a linear estimator of the form $\hat{f} = Ky + c$, where $K_{L \times |\mathcal{L}_m|}$ and c_L are the gain and offset of the estimator. Then, the best (minimum variance) linear unbiased estimator (BLUE estimator) is shown in [3] to produce the following estimation error covariance $E(H_{\mathcal{L}_m})$:

$$E(H_{\mathcal{L}_m}) = B(B^T H^T_{\mathcal{L}_m} \Sigma H_{\mathcal{L}_m} B)^{-1} B^T, \tag{17}$$

where $\Sigma_{\mathcal{L}_m \times \mathcal{L}_m}$ is the covariance matrix of sensor measurements, and $B_{L \times \eta}$ is an orthonormal basis of the null space of $\bar{P}$ and $\eta = \text{nullity}(\bar{P})$ such that $B^T B = I_\eta, BB^T = I_L$. Assuming that sensor noises are independent random variables, Σ is a diagonal matrix with each diagonal element being the variance of the sensor noise it belongs to. In order to obtain a scalar metric of how accurate an estimation is, $\text{trace}(E(H_{\mathcal{L}_m}))$ was considered as a cost function to be minimized in [3]. As a result, in this scenario, the sensor placement problem can be formulated as:

$$\begin{aligned} \underset{H_{\mathcal{L}_m} \subseteq H_{\mathcal{L}}}{\text{minimize}} \quad & \text{trace}(B(B^T H^T_{\mathcal{L}_m} \Sigma H_{\mathcal{L}_m} B)^{-1} B^T) \\ \text{subject to} \quad & |\mathcal{L}_m| \leq k. \end{aligned} \tag{18}$$

In [3], a simpler version of (18) with no constraint is solved via converting the problem into a continuous optimization problem. But, several thresholds and weighting parameters are required to be hand tuned for the continuous counterpart to lead to reasonable results. Herein, we prove that, using submodular properties, (18) can be easily solved without any hand tuning. Before proceeding, note that using the cyclic property of trace $(\text{trace}(AB) = \text{trace}(BA))$, and the fact that $B^T B = I$, we have that $\text{trace}(E(H_{\mathcal{L}_m})) = \text{trace}(B(B^T H^T_{\mathcal{L}_m} \Sigma H_{\mathcal{L}_m} B)^{-1} B^T) = \text{trace}(B^T H^T_{\mathcal{L}_m} \Sigma H_{\mathcal{L}_m} B)^{-1}$. We can also change minimization to maximization by maximizing the negative the of the estimation error covariance. Therefore, we can rewrite (18) as:

$$\begin{aligned} \underset{H_{\mathcal{L}_m} \subseteq H_{\mathcal{L}}}{\text{maximize}} \quad & -\text{trace}(B^T H^T_{\mathcal{L}_m} \Sigma H_{\mathcal{L}_m} B)^{-1} \\ \text{subject to} \quad & |\mathcal{L}_m| \leq k. \end{aligned} \tag{19}$$

The following theorem establishes the submodular property of (19).

Theorem 4. *Consider $\mathcal{V} = \{h_1, h_2, \cdots, h_L\}$ to be the set of possible rows of the matrix $H_{\mathcal{L}_m}$. For a subset $\mathcal{S} \subseteq \mathcal{V}$, let $H_{\mathcal{S}}$ be the H matrix constructed from the sensors being located according to the set $\mathcal{S}$. Then, the set function $u : 2^{\mathcal{V}} \to \mathbb{R}$ with $u(\mathcal{S})$ being $-trace(B^T H_{\mathcal{S}}^T \Sigma H_{\mathcal{S}} B)^{-1}$ is submodular.*

Before outlining the proof, we describe the following lemma from [12].

Lemma 1. *For arbitrary matrices $A \succ 0$ and $B \succeq 0$, define $X(t) = A + tB$, $t \geq 0$. Then, the continuously differentiable matrix functions of the form $\tilde{g} : \mathbb{R}_{\succ 0} \to \mathbb{R}$ are monotone decreasing if*

$$B \succeq 0 \Rightarrow \frac{d}{dt}\tilde{g}(X(t)) \leq 0, \quad \forall t.$$

Having Lemma 1 in mind and using proof ideas in [12], we can state the proof of Theorem 4.

Proof. Let $Q_{\mathcal{S}}$ be the matrix $B^T H_{\mathcal{S}}^T \Sigma H_{\mathcal{S}} B$. Construct the set function: $u_{h_i} : 2^{\mathcal{V} - h_i} \to \mathbb{R}$ as

$$u_{h_i}(\mathcal{S}) = -\text{trace}(Q_{\mathcal{S} \cup h_i}^{-1}) + \text{trace}(Q_{\mathcal{S}}^{-1}).$$

Moreover, consider the matrix functions $\tilde{u}_{h_i} : \mathbb{R}_{\succ 0} \to \mathbb{R}$ defined as $\tilde{u}_{h_i}(Q_{\mathcal{S}}) = u_{h_i}(\mathcal{S})$. It is easy to verify that if $\tilde{u}_{h_i}$ is monotone decreasing, then, u_{h_i} is also monotone decreasing and vice versa. Let Q_{h_i} be the Q matrix when $\mathcal{S} = h_i$. Define $X(t)$ as described in Lemma 1. Then, we will have:

$$\begin{aligned}
&\frac{d}{dt}\tilde{u}_{h_i}(X(t)) \\
&= \frac{d}{dt}\left(-\text{trace}(A + tB + Q_{h_i})^{-1} + \text{trace}(A + tB)^{-1}\right) \\
&= \text{trace}\left((A + tB + Q_{h_i})^{-2}B - (A + tB)^{-2}B\right).
\end{aligned}$$

The third line is simply achieved using the rules for derivatives of matrix inverses and the cyclic property of trace. Since $(A + tB + Q_{h_i}) \succeq (A + tB)$, we should have that:

$$(A + tB + Q_{h_i})^{-2}B - (A + tB)^{-2}B \preceq 0.$$

This implies that $\frac{d}{dt}(\tilde{u}_{h_i}) \leq 0$ (Trace of the product of a positive definite matrix and a negative semidefinite matrix is less than or equal to zero). Hence, using Lemma 1, $\tilde{u}_{h_i}$ is monotone decreasing. Therefore, u_{h_i} is monotone decreasing, and $-\text{trace}(Q_{\mathcal{S}}^{-1})$ is submodular using Theorem 1. □

An important requirement for employing (19) is that $Q_{H_{\mathcal{L}_m}} = B^T H_{\mathcal{L}_m}^T \Sigma H_{\mathcal{L}_m} B$ is invertible. Invertibility of $Q_{H_{\mathcal{L}_m}}$ imposes a constraint on the minimum number of sensors for obtaining a bounded objective function. Since $Q_{H_{\mathcal{L}_m}}$ is a $\eta \times \eta$ matrix, and η, the dimension of null space of $\bar{P}$, is in fact equal to the number of entry links, the minimum number of required sensors for having a bounded objective function is equal to the number of the entry links L_{entry}. Thus, the maximum allowed number of sensors in our budget constraint k must be greater than L_{entry}.

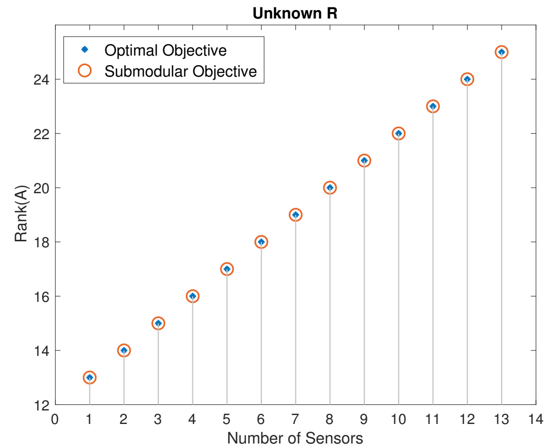

Fig. 2. Performance of the submodular optimization versus the true optimal solution for different budget constraints when turning ratios are unknown.

C. Practical Considerations

When using Algorithm 1, we start from an empty set of candidate locations, and at each iteration, simply add the location which leads to the greatest increase in the objective function. However, when starting with an empty set for solving (19), the matrix $Q_{H_{\mathcal{L}_m}}$ is not invertible until the η_{th} iteration. As a result, for the first η iterations, we can look for maximizing the rank of $Q_{H_{\mathcal{L}_m}}$ or maximizing $-\text{trace}(Q_{H_{\mathcal{L}_m}}^{+})$ as a proxy for the objective function of interest.

VI. Examples

To investigate the accuracy of the submodular approximation of problems (12) and (19), consider the grid network depicted in Figure 1 (similar to example network of [3]). The network has 12 nodes and 25 links. We probe the behavior of the submodular optimization for different budget constraints. The network size is small enough so that the true optimal solution of the problem can be found via exhaustive search. This enables us to compare the performance of the greedy algorithm with the true optimal performance. We consider the previous two scenarios.

When turning ratios are unknown, as mentioned earlier, we optimize for rank(A). Figure 3 demonstrates the objective function rank(A) for different budget constraints. As the figure shows, submodular optimization can find the optimal solution for every budget constraint. Note that in this case, there exist multiple optimal solutions as several configurations might lead to the same increase in rank(A).

Now, consider the case when R is known. An arbitrary profile of turning ratios is considered for R. We need to minimize the estimation error of the BLUE estimator defined in Section V. When starting the greedy algorithm, since invertibility of $Q_{\mathcal{H}_{\mathcal{L}_m}}$ has not been achieved yet, we seek for maximizing the rank of the cost function in (18), and among the solutions which lead to the same increase in rank, pick the one which minimizes $-\text{trace}Q_{\mathcal{H}_{\mathcal{L}_m}}^{+}$. We consider this criertia until invertiblity is attained. We ran exhaustive search

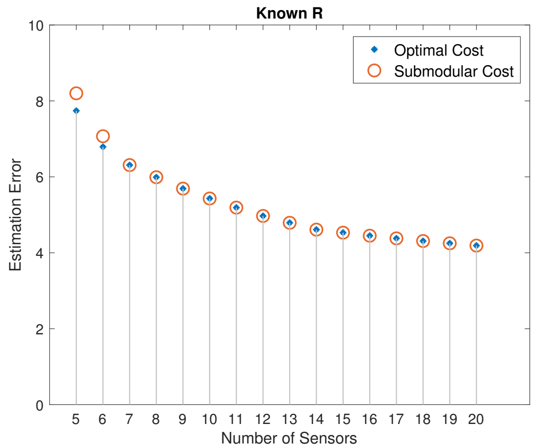

Fig. 3. Performance of the submodular optimization versus the true optimal solution for different budget constraints when turning ratios are known.

to find the true optimal solution of the problem. Figure 3 shows how submodular approach performs in comparison to the true optimal solution. As the Figure 3 shows, submodular optimization can successfully find the true optimal solution in most of the cases, and for the cases where the true solution is not found, the optimality gap is negligible.

VII. Conclusion and Future Work

We considered the problem of sensor placement for traffic networks. We examined two settings: no routing information is available versus the case that the turning ratios are available. We proved that in both cases, the problem has a submodular structure which allows for polynomial–time approximations of the formulated combinatorial optimization problem with guaranteed optimality bounds. We further demonstrated the capability of our approach in an example network. We are excited to observe that although sensor placement problem for traffic networks appears to be different from actuator/sensor placement for linear dynamical systems, it exhibits similar submodular behavior. An interesting direction to pursue is to examine submodularity of other formulations that take into account other aspects of the problem such as sensor failures and probabilistic events.

Acknowledgments

The authors would like to thank Linda Bushnell for useful discussions and her feedback on this work. This work is supported by the National Science Foundation under Grants CPS 1446145 and CPS 1545116.

References

[1] N. Mehr and R. Horowitz, "Probabilistic freeway ramp metering," *rN*, vol. 1, no. f1, p. 1, 2016.

[2] N. Mehr, D. Sadigh, R. Horowitz, S. S. Sastry, and S. A. Seshia, "Stochastic predictive freeway ramp metering from signal temporal logic specifications," in *American Control Conference (ACC), 2017*, pp. 4884–4889, IEEE, 2017.

[3] E. Lovisari, C. C. de Wit, and A. Kibangou, "Density/flow reconstruction via heterogeneous sources and optimal sensor placement in road networks," *Transportation Research Part C: Emerging Technologies*, vol. 69, pp. 451–476, 2016.

[4] A. Ehlert, M. G. Bell, and S. Grosso, "The optimisation of traffic count locations in road networks," *Transportation Research Part B: Methodological*, vol. 40, no. 6, pp. 460–479, 2006.

[5] S.-R. Hu and H.-T. Liou, "A generalized sensor location model for the estimation of network origin–destination matrices," *Transportation Research Part C: Emerging Technologies*, vol. 40, pp. 93–110, 2014.

[6] C. Antoniou, J. Barceló, M. Breen, M. Bullejos, J. Casas, E. Cipriani, B. Ciuffo, T. Djukic, S. Hoogendoorn, V. Marzano, *et al.*, "Towards a generic benchmarking platform for origin–destination flows estimation/updating algorithms: Design, demonstration and validation," *Transportation Research Part C: Emerging Technologies*, vol. 66, pp. 79–98, 2016.

[7] X. Zhou and G. F. List, "An information-theoretic sensor location model for traffic origin-destination demand estimation applications," *Transportation Science*, vol. 44, no. 2, pp. 254–273, 2010.

[8] X. Fei, H. S. Mahmassani, and P. Murray-Tuite, "Vehicular network sensor placement optimization under uncertainty," *Transportation Research Part C: Emerging Technologies*, vol. 29, pp. 14–31, 2013.

[9] A. Danczyk, X. Di, and H. X. Liu, "A probabilistic optimization model for allocating freeway sensors," *Transportation Research Part C: Emerging Technologies*, vol. 67, pp. 378–398, 2016.

[10] M. Shahabi and A. Unnikrishnan, "Robust hub network design problem," *Transportation Research Part E: Logistics and Transportation Review*, vol. 70, pp. 356–373, 2014.

[11] A. Krause, A. Singh, and C. Guestrin, "Near-optimal sensor placements in gaussian processes: Theory, efficient algorithms and empirical studies," *Journal of Machine Learning Research*, vol. 9, no. Feb, pp. 235–284, 2008.

[12] T. H. Summers, F. L. Cortesi, and J. Lygeros, "On submodularity and controllability in complex dynamical networks," *IEEE Transactions on Control of Network Systems*, vol. 3, no. 1, pp. 91–101, 2016.

[13] Z. Liu, A. Clark, P. Lee, L. Bushnell, D. Kirschen, and R. Poovendran, "Submodular optimization for voltage control," *IEEE Transactions on Power Systems*, 2017.

[14] A. Clark, P. Lee, B. Alomair, L. Bushnell, and R. Poovendran, "Combinatorial algorithms for control of biological regulatory networks," *arXiv preprint arXiv:1701.05273*, 2017.

[15] A. Krause and C. Guestrin, "Near-optimal observation selection using submodular functions," in *AAAI*, vol. 7, pp. 1650–1654, 2007.

[16] A. Clark, L. Bushnell, and R. Poovendran, "A supermodular optimization framework for leader selection under link noise in linear multi-agent systems," *IEEE Transactions on Automatic Control*, vol. 59, no. 2, pp. 283–296, 2014.

[17] G. L. Nemhauser, L. A. Wolsey, and M. L. Fisher, "An analysis of approximations for maximizing submodular set functions—i," *Mathematical Programming*, vol. 14, no. 1, pp. 265–294, 1978.

[18] A. Clark, B. Alomair, L. Bushnell, and R. Poovendran, *Submodularity in dynamics and control of networked systems*. Springer, 2016.

[19] F. L. Cortesi, T. H. Summers, and J. Lygeros, "Submodularity of energy related controllability metrics," in *Decision and Control (CDC), 2014 IEEE 53rd Annual Conference on*, pp. 2883–2888, IEEE, 2014.

[20] L. Lovász, "Submodular functions and convexity," in *Mathematical Programming The State of the Art*, pp. 235–257, Springer, 1983.

[21] P. Varaiya, "Max pressure control of a network of signalized intersections," *Transportation Research Part C: Emerging Technologies*, vol. 36, pp. 177–195, 2013.

[22] S. Koehler, N. Mehr, R. Horowitz, and F. Borrelli, "Stable hybrid model predictive control for ramp metering," in *Intelligent Transportation Systems (ITSC), 2016 IEEE 19th International Conference on*, pp. 1083–1088, IEEE, 2016.

[23] M. Wright and R. Horowitz, "Fusing loop and gps probe measurements to estimate freeway density," *IEEE Transactions on Intelligent Transportation Systems*, vol. 17, no. 12, pp. 3577–3590, 2016.

[24] F. Y. Tascikaraoglu, J. Lioris, A. Muralidharan, M. Gouy, and P. Varaiya, "Pointq model of an arterial network: calibration and experiments," *arXiv preprint arXiv:1507.08082*, 2015.