

Pricing Traffic Networks with Mixed Vehicle Autonomy

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Abstract—In a traffic network, vehicles normally select their routes selfishly. Consequently, traffic networks normally operate at an equilibrium characterized by Wardrop conditions. However, it is well known that equilibria are inefficient in general. In addition to the intrinsic inefficiency of equilibria, the authors recently showed that, in mixed-autonomy networks in which autonomous vehicles maintain a shorter headway than human-driven cars, increasing the fraction of autonomous vehicles in the network may increase the inefficiency of equilibria. In this work, we study the possibility of obviating the inefficiency of equilibria in mixed-autonomy traffic networks via pricing mechanisms. In particular, we study assigning prices to network links such that the overall or social delay of the resulting equilibria is minimum. First, we study the possibility of inducing such optimal equilibria by imposing a set of undifferentiated prices, i.e. a set of prices that treat both human-driven and autonomous vehicles similarly at each link. We provide an example which demonstrates that undifferentiated pricing is not sufficient for achieving minimum social delay. Then, we study differentiated pricing where the price of traversing each link may depend on whether vehicles are human-driven or autonomous. Under differentiated pricing, we prove that link prices obtained from the marginal cost taxation of links will induce equilibria with minimum social delay if the degree of road capacity asymmetry (i.e. the ratio between the road capacity when all vehicles are human-driven and the road capacity when all vehicles are autonomous) is homogeneous among network links.

I. INTRODUCTION

As autonomous vehicles become tangible technologies, studying their potential impact on transportation networks is becoming increasingly important. With the recent advances in deployment of autonomous vehicles, traffic networks will soon experience a transient era when both human-driven and autonomous vehicles will coexist on roads. Traffic networks with both vehicle types present on road, are referred to as traffic networks with mixed vehicle autonomy. The mobility and sustainability benefits of autonomous vehicles in networks with mixed autonomy have been investigated from different perspectives. Multiple studies have considered the problem of mitigating and damping shockwaves in traffic networks with mixed vehicle autonomy [1], [2], [3] via appropriate control of autonomous vehicles. In [4], intersection management strategies for mixed-autonomy networks are discussed. In [5], lane choices for autonomous vehicles are determined such that the overall performance of the system is optimized at traffic diverges.

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Connected and autonomous vehicles can group into vehicle platoons that are capable of maintaining a shorter headway while traveling. In [6], it was shown that as a consequence of shorter headways, vehicle platooning can lead to increases in road capacities, where up to three-fold increases in road capacity were achieved when all vehicles were assumed to be autonomous. The capacity increase that results from the presence of autonomous cars in mixed-autonomy networks was modelled in [7]. Throughout this paper, we assume that the headway that autonomous vehicle maintain is shorter or equal to that of human-driven vehicles and focus on the consequences of this difference.

Since vehicles select their routes selfishly, it is well known in the transportation literature that traffic networks tend to operate at equilibria characterized by Wardrop conditions, where vehicular flows are routed along the network paths such that no vehicle can gain any savings in its travel time by unilaterally changing its route [8]. However, it is known that due to the selfish behavior of vehicles, such network equilibria are in general inefficient. As an example, the well known Braess paradox [9] described an extreme scenario where adding a link to a network increased the overall network delay at equilibrium. Inefficiency of equilibria is commonly measured via the price of anarchy [10]. In [11], the price of anarchy of traffic networks with mixed vehicle autonomy was computed, and it was shown that the price of anarchy of networks with mixed vehicle autonomy is larger than that of networks with no autonomy. This implies that although the optimal overall delay of networks with mixed autonomy is lower (due to the capacity increase of autonomous vehicles), at equilibrium, the overall delay of networks with mixed autonomy is further from its optimum. Furthermore, in our previous work [12], [13], we showed that under constant vehicular demand, if the fraction of autonomous vehicles increases, the overall network delay may grow at equilibrium.

In this paper, we study how to cope with the inefficiency of equilibria in traffic networks with mixed vehicle autonomy such that the potential mobility benefits of autonomous vehicles are achieved, and minimum overall network delay also known as social delay is achieved at equilibrium. In particular, we study how to set prices on network links such that equilibria with minimum social delay are induced. Pricing has been extensively studied as a tool to create efficient equilibria in the previous literature (See [14], [15], [16], [17]). For traffic networks with only a single class of vehicles, a marginal cost taxation of network links was proposed in [18] which was proven to induce equilibria with minimum social delay. However, when there are multiple

classes of vehicles, traffic networks exhibit complex behavior such as nonuniqueness of equilibria [19]; thereby, prices that are obtained from marginal costs of network links may not be sufficient for inducing optimality of social delay at all possible equilibria [20], [21].

In this paper, we first study whether minimum social delay can be induced by setting undifferentiated prices on network links i.e. a set of prices that treat both human-driven and autonomous vehicles similarly at each link. We show through an example that undifferentiated prices are not in general enough for inducing minimum social delay at equilibrium. Then, we consider the setting where differentiated prices are assigned on network links, i.e. at every network link, the price assigned to human-driven vehicles can be different from that of autonomous vehicles. We prove that despite the fact that equilibrium is not necessarily unique in traffic networks with mixed autonomy, for traffic networks with a homogeneous degree of capacity asymmetry (networks where the ratio of the link capacities when all vehicles are human-driven over that all vehicles are autonomous is uniform throughout the network), with an appropriate set of prices, the social network delay of all induced equilibria is minimum. This is of paramount importance since uniqueness of social delay guarantees that regardless of which equilibrium the network operates at, the social delay will be minimum. In the absence of such a guarantee, the social delay may be optimal only for certain subsets of the induced equilibria.

This paper is organized as follows. In Section II, we describe how we model a traffic network with mixed autonomy. We review some relevant results from previous works in Section III. In Section IV, we discuss the limitations of undifferentiated pricing. In Section V, we demonstrate how differentiated pricing allows for inducing social optimality in certain traffic networks with mixed autonomy. Then, in Section VI, we conclude the paper and discuss future directions.

II. NONATOMIC ROUTING GAMES

We model a traffic network as a directed graph $G = (\mathcal{N}, \mathcal{L}, \mathcal{W})$, where \mathcal{N} is the set of network nodes, \mathcal{L} is the set of network links, and \mathcal{W} is the set of network origin destination (O/D) pairs. For each O/D pair $w \in \mathcal{W}$, r_w^h denotes the fixed given demand of human-driven cars, and r_w^a is the fixed given demand of autonomous cars that need to be routed along O/D pair w . Let $r = (r_w^h, r_w^a : w \in \mathcal{W})$ be the network vector of demands. We assume that the network topology is such that for each O/D pair $w \in \mathcal{W}$, there exists at least one path connecting its origin to its destination. We use \mathcal{P}_w to denote the set of all such paths connecting the origin to the destination in the O/D pair $w \in \mathcal{W}$. Moreover, we use $\mathcal{P} = \cup_{w \in \mathcal{W}} \mathcal{P}_w$ to denote the set of all network paths.

To represent flows along paths, for each O/D pair $w \in \mathcal{W}$ and path $p \in \mathcal{P}_w$, let f_p^h and f_p^a be the flows of human-driven and autonomous vehicles along path p , respectively. Note that each path $p \in \mathcal{P}$ connects one and only one O/D pair. Thus, the O/D pair associated to a path p is

determined unambiguously; hence, we do not carry the O/D pair associated to a path in our notation. For a path p , $f_p = f_p^h + f_p^a$ denotes the total flow along that path. Let $f = (f_p^h, f_p^a : p \in \mathcal{P})$ denote the vector of human-driven and autonomous flows along all network paths. A flow vector f is feasible for a given network G if for every O/D pair $w \in \mathcal{W}$, we have

$$\sum_{p \in \mathcal{P}_w} f_p^h = r_w^h, \quad (1a)$$

$$\sum_{p \in \mathcal{P}_w} f_p^a = r_w^a, \quad (1b)$$

$$\forall p \in \mathcal{P} : f_p^r \geq 0, f_p^a \geq 0. \quad (1c)$$

For link $l \in \mathcal{L}$ and a flow vector f , we use $f_l = \sum_{p \in \mathcal{P} : l \in p} f_p$ to denote the total flow of vehicles along link l . We further define $f_l^h = \sum_{p \in \mathcal{P} : l \in p} f_p^h$ and $f_l^a = \sum_{p \in \mathcal{P} : l \in p} f_p^a$ to be the total flow of human-driven and autonomous vehicles along link l , respectively. For a link $l \in \mathcal{L}$, we denote the delay per unit of flow that is incurred when traveling that link by the link delay function $e_l : \mathbb{R}^2 \rightarrow \mathbb{R}$. For traffic networks with mixed vehicle autonomy, using the well known US Bureau of Public Roads (BPR) [22] delay functions and the capacity model of [7], the delay of traversing each link $l \in \mathcal{L}$ will be of the following form (See [12] for details)

$$e_l(f_l^h, f_l^a) = a_l + \gamma_l \left(\frac{f_l^h}{m_l} + \frac{f_l^a}{M_l} \right)^{\beta_l}, \quad (2)$$

where a_l and γ_l are positive link parameters, β_l is a positive integer, m_l is the capacity of link l when all vehicles are human-driven, and M_l is the capacity of link l when all vehicles are autonomous. Following [11], for each link $l \in \mathcal{L}$, we define

$$\mu_l := \frac{m_l}{M_l}, \quad (3)$$

to be the degree of capacity asymmetry along link l . Since autonomous vehicles are capable of maintaining a shorter headway, it is assumed that for every link $l \in \mathcal{L}$, $M_l \geq m_l$; thus, $\mu_l \leq 1$. If the degree of capacity asymmetry is the same among all network links, i.e. for every link $l \in \mathcal{L}$, $\mu_l = \mu$, then, the network is said to have a homogeneous degree of capacity asymmetry μ .

Since the delay functions are additive, we can define delay along a path $p \in \mathcal{P}$ to be

$$e_p(f) := \sum_{l \in \mathcal{L} : l \in p} e_l(f_l^h, f_l^a). \quad (4)$$

Note that for each path $p \in \mathcal{P}$, the delay along path p depends on the whole flow vector f rather than just f_p^h and f_p^a since the network links are shared among vehicles along different paths. We define the network *overall delay* (also known as *social delay*) to be

$$J(f) = \sum_{p \in \mathcal{P}} f_p e_p(f). \quad (5)$$

We say that a flow vector $f^* = (f_p^{h*}, f_p^{a*} : p \in \mathcal{P})$ is socially optimal when it minimizes $J(f)$ subject to relations (1). The optimal social delay is denoted by J^* . In this paper, we assume that a network pricing infrastructure is in place in which vehicles are charged as they travel along specific links. Moreover, vehicles may be charged differently, depending on whether they are human driven or autonomous. When the network links are priced, for each link $l \in \mathcal{L}$, we use $\tau_l^h \geq 0$ and $\tau_l^a \geq 0$ to denote the price for human-driven and autonomous vehicles along link l , respectively. Let $\tau := (\tau_l^h, \tau_l^a : l \in \mathcal{L})$ be the vector of link prices. The price of human-driven vehicles along a path $p \in \mathcal{P}$ is defined as $\tau_p^h := \sum_{l \in \mathcal{L}: l \in p} \tau_l^h$. Likewise, define $\tau_p^a := \sum_{l \in \mathcal{L}: l \in p} \tau_l^a$ to be the price of autonomous vehicles along path p .

A common assumption in the transportation literature is that every O/D pair consists of infinitesimally small agents that select their routes selfishly. Thus, every agent selects a route from its origin to its destination such that its own travel delay is minimized. As a result, traffic networks achieve an equilibrium, where no agent has an incentive for unilaterally changing its route. A traffic network is at equilibrium if well known Wardrop conditions hold [8].

Note that when prices are set, each agent is subjected to both travel time and monetary costs. Hence, for traversing a path $p \in \mathcal{P}$, an agent experiences a delay $e_p(f)$ and pays a price equal to either τ_p^h or τ_p^a , depending on whether it is a human-driven or an autonomous vehicle. Thus, assuming that all agents value travel delays and monetary costs identically, the cost of an agent along a path p is either $e_p(f) + \tau_p^h$ or $e_p(f) + \tau_p^a$ depending on whether the agent is human-driven or autonomous. We define the link traversal cost functions c_l^h and c_l^a to be the following

$$c_l^h(f_l^h, f_l^a) := e_l(f_l^h, f_l^a) + \tau_l^h, \quad (6a)$$

$$c_l^a(f_l^h, f_l^a) := e_l(f_l^h, f_l^a) + \tau_l^a. \quad (6b)$$

Similarly, we define the cost of traversing a path $p \in \mathcal{P}$ for human-driven and autonomous vehicles to respectively be

$$c_p^h(f) := e_p(f) + \tau_p^h, \quad (7a)$$

$$c_p^a(f) := e_p(f) + \tau_p^a. \quad (7b)$$

For a given vector of link prices τ , we define a nonatomic selfish routing to be the triple (G, r, c) .

Remark 1. For every link $l \in \mathcal{L}$, we use the term link cost for human-driven or autonomous vehicles to refer to $c_l^h(f_l^h, f_l^a)$ or $c_l^a(f_l^h, f_l^a)$, respectively. However, we use the term link delay to refer solely to $e_l(f_l^h, f_l^a)$, which is the delay of travel along link l , excluding the corresponding price. Note that the cost of traversing a link $l \in \mathcal{L}$ might be different for human-driven and autonomous vehicles, while the delay of traversing link l is the same for both classes of vehicles.

Remark 2. When a price vector τ is set, although the traversal cost of a link perceived by every agent may be different from the delay of travel along that link, the overall performance of the system is still measured via the overall delay incurred by all agents. The goal of this work is to find

link prices such that the overall delay of the system perceived by the society is minimized.

The overall cost of a routing game (G, r, c) is defined by

$$C(f) = \sum_{p \in \mathcal{P}} f_p^h c_p^h(f) + f_p^a c_p^a(f). \quad (8)$$

For a priced network, Wardrop equilibria are defined via the following.

Definition 1. For a routing game (G, r, c) , a feasible flow vector $f = (f_p^h, f_p^a : p \in \mathcal{P})$ is an equilibrium if and only if for every O/D pair $w \in \mathcal{W}$ and every pair of paths $p, p' \in \mathcal{P}_w$, we have

$$f_p^h (c_p^h(f) - c_{p'}^h(f)) \leq 0, \quad (9a)$$

$$f_p^a (c_p^a(f) - c_{p'}^a(f)) \leq 0. \quad (9b)$$

Remark 3. In general, despite the classical setting of a single vehicle class where the Wardrop equilibrium is unique [23], in our mixed-autonomy setting, there may exist multiple Wardrop equilibria satisfying (9).

Notice that equations (9) imply that if for an O/D pair $w \in \mathcal{W}$, and two paths $p, p' \in \mathcal{P}_w$, the flows f_p^h and $f_{p'}^h$ are nonzero, we have $c_p^h(f) = c_{p'}^h(f)$ (we can argue similarly for autonomous vehicles). Moreover, if at equilibrium, the flow along a path is zero, its travel cost cannot be smaller than that of the other paths with nonzero flow of the same vehicle class. Therefore, we can define the following.

Definition 2. For a routing game (G, r, c) , if f is an equilibrium flow vector, for each O/D pair $w \in \mathcal{W}$, define the cost of travel for human-driven and autonomous vehicles to respectively be

$$c_w^h(f) = \min_{p \in \mathcal{P}_w} c_p^h(f), \quad (10a)$$

$$c_w^a(f) = \min_{p \in \mathcal{P}_w} c_p^a(f). \quad (10b)$$

Since at equilibrium, for each O/D pair w and each class of vehicles, the cost of travel along the paths that have nonzero flow of that class is the same and equal to cost of travel for that class, we have

$$C(f) = \sum_{w \in \mathcal{W}} r_w^h c_w^h(f) + r_w^a c_w^a(f). \quad (11)$$

III. PRIOR WORK

In this section, we review some results from the previous literature that we will further use in this paper. The following proposition is a generalization of the results in [20].

Proposition 1. For a routing game (G, r, c) , let f^* be an optimizer of the network social delay J , and J^* be the minimum social delay of the network. If for each link $l \in \mathcal{L}$, link prices τ is set to be

$$\tau_l^h = (f_l^{h*} + f_l^{a*}) \left(\frac{\partial}{\partial f_l^h} e_l(f_l^h, f_l^a) \right) \Big|_{f^*}, \quad (12a)$$

$$\tau_l^a = (f_l^{h*} + f_l^{a*}) \left(\frac{\partial}{\partial f_l^a} e_l(f_l^h, f_l^a) \right) \Big|_{f^*}, \quad (12b)$$

then, there exists at least one equilibrium flow vector f for the routing game (G, r, c) such that the network social delay is optimal at this equilibrium, i.e. $J(f) = J^*$.

Proof. It is easy to verify that (12) renders f^* an equilibrium flow vector by verifying the KKT conditions at the optimal point f^* . For completeness, we have included the proof of Proposition 1 in Appendix I. \square

Note that Proposition 1 indicates that if prices are obtained from (12), the social delay of one of the induced equilibria is optimal. However, the social delay of other equilibria may not necessarily be optimal.

As mentioned previously, in general, the equilibrium is not unique in our mixed-autonomy setting. However, we use the following result from [24] in the remainder of the paper to establish some properties of equilibria in the mixed-autonomy setting.

Proposition 2. For a routing game (G, r, c) , if along each link $l \in \mathcal{L}$, the link traversal cost functions c_l^h and c_l^a are strictly increasing functions of the total flow along that link $f_l = f_l^h + f_l^a$, and the link cost functions c_l^h and c_l^a are identical up to additive constants, then, at equilibrium, the total flow along each link $l \in \mathcal{L}$ is unique.

It is important to mention that in our mixed-autonomy setting, since the link cost functions (6) depend on the flow of each vehicle class, not the total flow along the link, Proposition 2 does not directly apply to our setting. Nevertheless, we will further apply Proposition 2 to an auxiliary routing game to obtain some of our results.

IV. UNDIFFERENTIATED PRICES

When it comes to setting prices for network links, generally, an ideal set of prices is the one that induces an equilibrium flow vector that minimizes the network social delay. When there are multiple classes of vehicles in a network, for instance human-driven and autonomous vehicles in our scenario, it is important to determine whether it is possible to induce an equilibrium that optimizes social delay via prices that do not differentiate vehicle classes. This is of practical significance because undifferentiated prices are much easier to implement. Unfortunately, in this section, we show through a counterexample that, for traffic networks with mixed autonomy, it is not always possible to induce equilibrium flows with minimum social delay by simply applying undifferentiated prices, even in networks with a homogeneous degree of capacity asymmetry.

Example 1. Consider the network shown in Figure 1. Assume that the network has a homogeneous degree of capacity asymmetry $\mu = \frac{1}{3}$. There are two O/D pairs $\mathcal{W} = \{AB, AC\}$. For each O/D pair, there are two possible paths. The demand of O/D pairs are $r_{AB}^h = 7.5$, $r_{AB}^a = 4.5$, $r_{AC}^h = 1.2$, and $r_{AC}^a = 4.8$. For every link $l, 1 \leq l \leq 4$, assume that $\gamma_l = 1$, and $\beta_l = 1$. The other link parameters are $a_1 = 9, a_2 = 3, a_3 = 0.6$, and $a_4 = 0.6$ while $m_1 = 3, m_2 = 0.5, m_3 = 0.7$, and $m_4 = 0.5$. Note that

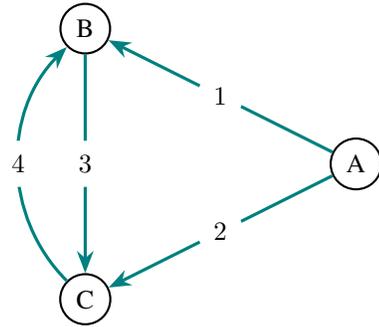


Fig. 1: A network with two O/D pairs from A to B and A to C.

for each link l , the parameter M_l is determined through the relation $M_l = m_l/\mu$. For this network, it is easy to compute the optimal social delay, which is $J^* = 193.54$. In order to see whether a set of undifferentiated prices can achieve this optimal social delay, we solve the following optimization problem for this network

$$\begin{aligned} \min \quad & \sum_{p \in \mathcal{P}} (f_p^h + f_p^a) e_p(f) \\ \text{subject to} \quad & \text{Equations (1),} \\ & \text{Equations (9),} \\ & \forall l \in \mathcal{L} : \tau_l^h = \tau_l^a \geq 0, \end{aligned} \quad (13)$$

Note that if the minimum social delay J^* can be achieved via undifferentiated prices, the optimal value of optimization problem (13) must be equal to J^* . However, the minimum value of (13) for the network in Figure 1 is 195.597 which is clearly greater than J^* . This indicates that an equilibrium with socially optimal delay cannot necessarily be induced by undifferentiated link prices in general.

V. DIFFERENTIATED PRICES

Having shown in Example 1 that in general, undifferentiated pricing cannot induce an equilibrium with socially optimal delay, a natural question is whether *differentiated* prices can be employed to induce an equilibrium with minimum social delay. In other words, if at every link $l \in \mathcal{L}$, we allow τ_l^h and τ_l^a to be different, does there exist a price vector $\tau = (\tau_l^h, \tau_l^a : l \in \mathcal{L})$ that induces equilibria with minimum social delay? In this section, we prove that such differentiated prices exist, and we find them.

A. Homogeneous Networks

The following theorem establishes the existence of optimal prices for traffic networks with a homogeneous degree of capacity asymmetry and also provides a recipe for how to find the optimal price values.

Theorem 1. Consider a routing game (G, r, c) with a homogeneous degree of capacity asymmetry μ . Let f^* and J^* be an optimizer and the minimum value of the social delay, respectively. Then, for each link $l \in \mathcal{L}$, if the link

prices are set to be

$$\tau_l^h = (f_l^{h*} + f_l^{a*}) \left(\frac{\partial}{\partial f_l^h} e_l(f_l^h, f_l^a) \right) \Big|_{f^*}, \quad (14a)$$

$$\tau_l^a = (f_l^{h*} + f_l^{a*}) \left(\frac{\partial}{\partial f_l^a} e_l(f_l^h, f_l^a) \right) \Big|_{f^*}, \quad (14b)$$

then, all induced equilibria of the game (G, r, c) have the same social delay, which is equal to J^* .

Proof. For the routing game (G, r, c) where the link prices are obtained via (14), Proposition 1 implies that there exists one equilibrium with minimum social delay J^* . We prove that when the above prices are set, all induced equilibria of (G, r, c) have the same social delay. This would then imply that all induced equilibria of the game (G, r, c) have the unique social delay J^* , as was claimed.

Therefore, it remains to prove uniqueness of social delay at equilibria of (G, r, c) when link prices are obtained from (14). In order to do so, we construct an auxiliary game instance $(G, \tilde{r}, \tilde{c})$, with the same network graph G and O/D pairs \mathcal{W} , where the demand of O/D pairs, link traversal delays and cost functions are defined as follows. For each O/D pair $w \in \mathcal{W}$, define the demand of human-driven and autonomous cars \tilde{r}_w^h and \tilde{r}_w^a in the auxiliary game to be

$$\tilde{r}_w^h := r_w^h, \quad (15a)$$

$$\tilde{r}_w^a := \mu r_w^a. \quad (15b)$$

Moreover, for every link $l \in \mathcal{L}$, let the link delay functions of the auxiliary game $(G, \tilde{r}, \tilde{c})$ be defined as

$$\tilde{e}_l(\tilde{f}_l^h, \tilde{f}_l^a) := a_l + \gamma_l \left(\frac{\tilde{f}_l^h + \tilde{f}_l^a}{m_l} \right)^{\beta_l}. \quad (16)$$

Additionally, for every link $l \in \mathcal{L}$, with the prices as in (14), define the link cost functions in the auxiliary game to be

$$\tilde{c}_l^h(\tilde{f}_l^h, \tilde{f}_l^a) := \tilde{e}_l(\tilde{f}_l^h, \tilde{f}_l^a) + \tau_l^h, \quad (17a)$$

$$\tilde{c}_l^a(\tilde{f}_l^h, \tilde{f}_l^a) := \tilde{e}_l(\tilde{f}_l^h, \tilde{f}_l^a) + \tau_l^a. \quad (17b)$$

Now, let $f = (f_p^h, f_p^a : p \in \mathcal{P})$ be an equilibrium of the original game (G, r, c) . For every path $p \in \mathcal{P}$, define $\tilde{f}_p^h := f_p^h$ and $\tilde{f}_p^a := \mu f_p^a$. We claim that $\tilde{f} = (\tilde{f}_p^h, \tilde{f}_p^a : p \in \mathcal{P})$ is an equilibrium flow vector for the auxiliary game $(G, \tilde{r}, \tilde{c})$. It can be easily verified that for every origin-destination pair $w \in \mathcal{W}$, we have $\sum_{p \in \mathcal{P}_w} \tilde{f}_p^h = \tilde{r}_w^h$ and $\sum_{p \in \mathcal{P}_w} \tilde{f}_p^a = \tilde{r}_w^a$. Thus, \tilde{f} is a feasible flow vector for the auxiliary game. Moreover, it is easy to see that for every link $l \in \mathcal{L}$, we have $\tilde{f}_l^a = \mu f_l^a$; therefore, using the definition of \tilde{f} and Equations (2) and (16), for every link $l \in \mathcal{L}$, we establish

the following

$$\begin{aligned} \tilde{e}_l(\tilde{f}_l^h, \tilde{f}_l^a) &= a_l + \gamma_l \left(\frac{\tilde{f}_l^h + \tilde{f}_l^a}{m_l} \right)^{\beta_l} \\ &= a_l + \gamma_l \left(\frac{f_l^h + \mu f_l^a}{m_l} \right)^{\beta_l} \\ &= a_l + \gamma_l \left(\frac{f_l^h}{m_l} + \frac{f_l^a}{M_l} \right)^{\beta_l} \\ &= e_l(f_l^h, f_l^a). \end{aligned} \quad (18)$$

Thus, from (6), (17), and (18), for every link $l \in \mathcal{L}$, we have

$$\tilde{c}_l^h(\tilde{f}_l^h, \tilde{f}_l^a) = c_l^h(f_l^h, f_l^a) \quad (19a)$$

$$\tilde{c}_l^a(\tilde{f}_l^h, \tilde{f}_l^a) = c_l^a(f_l^h, f_l^a). \quad (19b)$$

Now since f is an equilibrium for (G, r, c) , using (9) and (19), for every O/D pair $w \in \mathcal{W}$ and pair of paths $p, p' \in \mathcal{P}_w$, we have

$$f_p^h \left(\tilde{c}_p^h(\tilde{f}) - \tilde{c}_{p'}^h(\tilde{f}) \right) \leq 0, \quad (20a)$$

$$f_p^a \left(\tilde{c}_p^a(\tilde{f}) - \tilde{c}_{p'}^a(\tilde{f}) \right) \leq 0. \quad (20b)$$

Multiplying (20b) by the positive constant μ , we have

$$f_p^h \left(\tilde{c}_p^h(\tilde{f}) - \tilde{c}_{p'}^h(\tilde{f}) \right) \leq 0, \quad (21a)$$

$$\mu f_p^a \left(\tilde{c}_p^a(\tilde{f}) - \tilde{c}_{p'}^a(\tilde{f}) \right) \leq 0. \quad (21b)$$

Using the definition of \tilde{f} , from (21) we can conclude the following

$$\tilde{f}_p^h \left(\tilde{c}_p^h(\tilde{f}) - \tilde{c}_{p'}^h(\tilde{f}) \right) \leq 0, \quad (22a)$$

$$\tilde{f}_p^a \left(\tilde{c}_p^a(\tilde{f}) - \tilde{c}_{p'}^a(\tilde{f}) \right) \leq 0. \quad (22b)$$

Note that these are precisely the equilibrium conditions for the auxiliary game $(G, \tilde{r}, \tilde{c})$, which proves our claim that \tilde{f} is an equilibrium for the auxiliary game.

Now, note that the conditions of Proposition 2 hold for the auxiliary game $(G, \tilde{r}, \tilde{c})$ since for every link $l \in \mathcal{L}$, the link traversal costs of human-driven and autonomous cars \tilde{c}_l^h and \tilde{c}_l^a are strictly increasing functions of the total link flow $\tilde{f}_l = \tilde{f}_l^h + \tilde{f}_l^a$. Moreover, motivated by (17), the costs of human-driven and autonomous cars are identical up to a constant. Thus, using Proposition 2 for $(G, \tilde{r}, \tilde{c})$, for every link $l \in \mathcal{L}$, the total link flow $\tilde{f}_l = \tilde{f}_l^h + \tilde{f}_l^a$ is unique among all the equilibria. Therefore, using the definition of \tilde{f} , the fact that \tilde{f} is an equilibrium flow vector for $(G, \tilde{r}, \tilde{c})$, and the connection between f and \tilde{f} , we conclude that at every link $l \in \mathcal{L}$, we must have that $f_l^h + \mu f_l^a$ is unique for all equilibria of (G, r, c) . Additionally, from (16) and (17), for every link $l \in \mathcal{L}$, uniqueness of the total link flow at equilibrium in the auxiliary game implies that the link traversal costs $\tilde{c}_l^h(\tilde{f}_l^h, \tilde{f}_l^a)$ and $\tilde{c}_l^a(\tilde{f}_l^h, \tilde{f}_l^a)$ are unique. Hence, from (19), we can conclude that in (G, r, c) , for each link $l \in \mathcal{L}$, the link traversal costs $c_l^h(f_l^h, f_l^a)$ and $c_l^a(f_l^h, f_l^a)$ are also unique for all equilibrium flow vectors f . Thus,

for each O/D pair $w \in \mathcal{W}$, (10) results in uniqueness of travel costs of both human-driven and autonomous cars c_w^h and c_w^a . Consequently, from (11), we realize that the overall cost $C(f)$ is unique for all equilibrium flow vectors f of (G, r, c) .

Now, using (4), (6), and (8), we can rewrite the social cost of (G, r, c) as

$$C(f) = \sum_{l \in \mathcal{L}} f_l^h c_l^h(f) + f_l^a c_l^a(f) \quad (23)$$

$$= \sum_{l \in \mathcal{L}} (f_l^h + f_l^a) e_l^h(f) + f_l^h \tau_l^h + f_l^a \tau_l^a \quad (24)$$

$$= J(f) + \sum_{l \in \mathcal{L}} f_l^h \tau_l^h + f_l^a \tau_l^a. \quad (25)$$

Notice that under homogeneity of the network, using the special structure of (2), it is easy to see that Equation (14) implies that for every link $l \in \mathcal{L}$, we have

$$\tau_l^a = \mu \tau_l^h. \quad (26)$$

Substituting (26) in (25), we have

$$C(f) = J(f) + \sum_{l \in \mathcal{L}} \tau_l^h (f_l^h + \mu f_l^a). \quad (27)$$

Note that using our introduced auxiliary game, we proved that the overall cost $C(f)$ is unique for all equilibrium flow vectors f . Furthermore, we proved that for every link l , $f_l^h + \mu f_l^a = \tilde{f}_l^h + \tilde{f}_l^a$ is unique for all equilibria. Hence, from (27), we can conclude that the social delay $J(f)$ is also unique for all equilibrium flow vectors. This completes our proof. \square

Note that if for each link $l \in \mathcal{L}$, the link prices are obtained from (14), since the link delay function (2) is increasing in the flow of each vehicle class, the prices that result from (2) are always nonnegative which is in accordance with our initial assumption. The link prices obtained by (14) are in fact the extra term in the marginal cost of each vehicle class.

B. Heterogeneous Networks

If the road degree of capacity asymmetry is not homogeneous, but a central authority sets link prices to be obtained from (14), there still exists at least one induced equilibrium flow vector that achieves the minimum social delay. However, for heterogeneous networks, the social delay is not necessarily unique. Therefore, although the social delay of one induced equilibrium is optimal, the social delay of other induced equilibria might not be optimal. For such networks, optimally of the social delay may not be achieved in all induced equilibria by setting the prices to be obtained from (14).

VI. CONCLUSION AND FUTURE WORK

We considered the problem of inducing efficient equilibria in traffic networks with mixed vehicle autonomy via pricing. We showed that minimum social delay may not be attained by imposing undifferentiated link prices, in which human-driven and autonomous vehicles are treated identically. Then, we proved that in mixed-autonomy traffic networks with a

homogeneous degree of capacity asymmetry, if differentiated prices are allowed, which treat human-driven and autonomous vehicles differently, link prices can be determined such that all induced equilibria have minimum social delay. For future steps, it is interesting to study path-based price collection to achieve further objectives such as collecting the minimum possible monetary value or obtaining a fair pricing policy. It is also important to study the existence of prices when users are heterogeneous, i.e. users that might value the monetary costs of prices differently.

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APPENDIX I

Since the social delay defined by (5) is a continuous function of flows along network paths, and the set of feasible flows satisfying (1) is compact, there exists a flow vector f^* that optimizes the social delay, i.e. f^* is the optimizer of the following optimization problem

$$\begin{aligned} \min_f \quad & J(f) \\ \text{subject to} \quad & \forall p \in \mathcal{P} : f_p^h \geq 0, f_p^a \geq 0, \\ & \forall w \in \mathcal{W} : \sum_{p \in \mathcal{P}_w} f_p^h = r_w^h, \sum_{p \in \mathcal{P}_w} f_p^a = r_w^a. \end{aligned} \quad (28)$$

Note that all constraints in (28) are affine functions of the decision variable f . Therefore, the regularity conditions hold for (28) (see, for instance, Theorem 5.1.3 and Lemma 5.1.4 in [25]). Thus, the optimizer f^* must satisfy the KKT conditions. For each path $p \in \mathcal{P}$, let $\lambda_p^h \geq 0$ and $\lambda_p^a \geq 0$ be the Lagrange multipliers associated with the nonnegativity constraint of the flows of human-driven and autonomous vehicles along path p , respectively. Similarly, for each O/D pair $w \in \mathcal{W}$, let ν_w^h and ν_w^a be the Lagrange multipliers associated with the flow conservation constraints for human-driven and autonomous vehicles along the O/D pair w , respectively. Then, for a fixed path $p \in \mathcal{P}_w$ associated to an O/D pair $w \in \mathcal{W}$, for the flow of human-driven cars, the stationarity condition imposes the following

$$\left. \frac{\partial}{\partial f_p^h} J(f) \right|_{f^*} = \lambda_p^h - \nu_w^h. \quad (29)$$

From (4), (5), and (14), we have

$$\begin{aligned} \left. \frac{\partial}{\partial f_p^h} J(f) \right|_{f^*} &= \\ & \sum_{l \in \mathcal{L}: l \in p} \left(e_l(f_l^h, f_l^a) + (f_l^h + f_l^a) \frac{\partial}{\partial f_l^h} e_l(f_l^h, f_l^a) \right) \Big|_{f^*} \\ &= e_p(f^*) + \tau_p^h. \end{aligned}$$

Using this together with (29), we can conclude that for every path $p \in \mathcal{P}$, we have

$$e_p(f^*) + \tau_p^h = \lambda_p^h - \nu_w^h. \quad (30)$$

On the other hand, complementary slackness requires that for every path $p \in \mathcal{P}$

$$\lambda_p^h f_p^h = 0. \quad (31)$$

Now, for a fixed O/D pair $w \in \mathcal{W}$, consider a pair of paths $p, p' \in \mathcal{P}_w$. If $f_p^h > 0$, from (31), we must have $\lambda_p^h = 0$. Then, from (30) and nonnegativity of $\lambda_{p'}^h$, we have

$$e_p(f^*) + \tau_p^h = -\nu_w^h \leq \lambda_{p'}^h - \nu_w^h = e_{p'}(f^*) + \tau_{p'}^h, \quad (32)$$

where in the last equality, we have used (30) for the path p' . Similarly, for autonomous cars along the two paths p and p' , if $f_p^a > 0$, we must have

$$e_p(f^*) + \tau_p^a = -\nu_w^a \leq \lambda_{p'}^a - \nu_w^a = e_{p'}(f^*) + \tau_{p'}^a. \quad (33)$$

Note that (32) implies (9a), the reason being that if $f_p^h = 0$, (9a) automatically holds, and if $f_p^h > 0$, (9a) holds since as we showed above, $c_p^h(f^*) \leq c_{p'}^h(f^*)$. Likewise, (33) implies (9b). Hence, once prices are set according to (14), the optimal flow f^* is a Wardrop equilibrium for the game (G, r, c) ; thus, there exists at least one induced equilibrium with minimum social delay once prices are obtained from (14).

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