Pergamon

Transpn Res.-C, Vol. 5, No. 1, pp. 11-37, 1997

0968-090X/97 $17.00 + 0.00

PII: S0968-090X(96)00024-1

AN AUTOMATED HIGHWAY SYSTEM LINK LAYER CONTROLLER FOR TRAFFIC FLOW STABILIZATION[1]

PERRY Y. LI,[†] ROBERTO HOROWITZ,* LUIS ALVAREZ, JONATHAN FRANKEL[‡] and ANNE M. ROBERTSON[§]

Department of Mechanical Engineering, University of California at Berkeley, Berkeley, California 94720, U.S.A.

(Received 21 November 1995; in revised form 31 October 1996)

Abstract—Controls for the link layer in the Automated Highway System (AHS) hierarchy proposed in the Partners for Advanced Transit and Highways (PATH) are developed. The link layer is modeled using vehicle conservation flow models. Desired traffic conditions on the highway are given by a pair of density and velocity profiles which are assumed to be consistent with the demand on, and the capability of, the highway system. The link layer control laws presented in this paper then stabilize the actual traffic condition to the desired values. Control laws are derived for three highway topologies: a single lane highway, a highway with multiple discrete lanes and a two-dimensional highway with an arbitrary flow pattern. The control laws obtained for each of the topologies is distributed and are suited for implementation in the lower levels of the AHS control hierarchy. Simulation results are also presented. © 1997 Elsevier Science Ltd

INTRODUCTION

To improve the capacity and safety of existing highways, the concept of Automated Highway Systems (AHS) has been proposed (Varaiya, 1993). The key to improving capacity and safety is platooning: the organization of vehicles into closely spaced groups. The spacing between these platoons is large, but platooning decreases the mean inter-vehicle distance and thus allows the highway to accommodate more cars. Because of the small distance between cars in a platoon, the relative impact velocity (and hence the impact energy) is small if the vehicles do collide. For platooning to work, vehicles must be automated. Human drivers cannot react quickly enough to follow each other in close proximity at high speed.

The current AHS architecture proposed in the California Partners for Advanced Transit and Highways (PATH) (Varaiya, 1993) consists of five hierarchical layers (Fig. 1): the network layer, the link layer, the planning and coordination layer, the regulation layer and the physical layer.

There is a different control objective for each of these layers. The network layer routes vehicles within the network of highways so that on average vehicles entering the system reach their destinations in the shortest amount of time. The link layer controller establishes traffic conditions, which can be defined by density and flow profiles, so as to realize the capacity of a single highway or a stretch of highway. Vehicles on the automated highway can engage in a finite number of activities. They can be the leader of a platoon, a follower in a platoon or a leader executing a special maneuver. In the current design for normal mode operation, the maneuvers are join, split and change lane, which correspond to the vehicle performing the maneuver of joining a platoon, leaving a platoon and changing lanes. A larger set of maneuvers will be necessary in degraded mode operation, when extraordinary situations must be considered (Lygeros *et al.*, 1995). The coordination layer determines when a vehicle should perform one of these maneuvers and ensures that the maneuver occurs in an orderly and safe fashion. This is achieved by the

[1]Research supported by UCB-ITS PATH grant MOU-135.

*Author for correspondence.

[†]Department of Mechanical Engineering, University of Minnesota, Minneapolis, MN 55455, U.S.A.

[‡]Currently at Applied Materials, 350 Bowers Avenue, Santa Clara, California, U.S.A.

[§]Currently at Department of Mechanical Engineering, University of Pittsburgh, Pittsburgh, Pensylvania 15261, U.S.A.

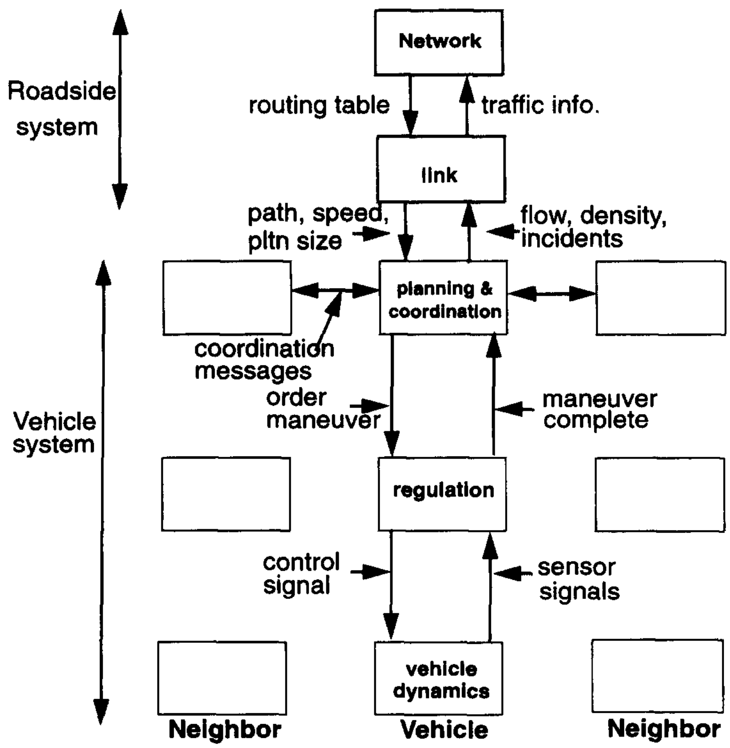


Fig. 1. Hierarchial architecture of AHS in the PATH project.

structured communications between vehicles (Varaiya, 1993). Once the maneuvers are chosen, the regulation layer control laws carry out the maneuvers (Frankel *et al.*, 1995; Godbole and Lygeros, 1993).

Much work has been done developing and verifying controllers for the regulation and coordination layers. Readers are referred to (Varaiya, 1993; Frankel *et al.*, 1995; Godbole *et al.*, 1994; Godbole and Lygeros, 1993) for further details. Control strategies at the macroscopic level for the link layer are more scarce. We review some of the relevant ones here.

In Karaaslan *et al.* (1990), the authors present a detailed traffic flow model based on the behavior of human drivers. They argue that the dynamics caused by the human driver behavior prevent full utilization of the highway in the presence of congestion. With cars under automatic control this need not be the case. It is proposed that if one of the terms that describe driver behavior is replaced with a control term intended to homogenize the density profile, then the capacity of the highway can be better realized. This problem can be considered one of tracking a uniform density profile.

Chien *et al.* (1993) generalize this problem to the tracking of an arbitrary density profile. Using a macroscopic traffic model similar to that in Karaaslan *et al.* (1990), they derive a controller that commands a desired velocity at each section of the highway such that the density of the entire highway conforms to a specified density profile. Their model is based on the behavior of human drivers. While it is possible to design control laws for automated vehicles so that they behave like those driven by people, it is not necessary. Moreover, the developed control law is based upon the inversion of the traffic flow dynamics, which requires a certain traffic flow controllability condition. This condition is violated when the density in any section of the highway becomes too small. The control action at a point in the highway requires information from the entire highway. This problem is alleviated by a dynamic version of the control law that solves the matrix inversion dynamically. No multiple lane or lane change commands are considered in this work.

A description of a link layer controller consistent with the AHS architecture (Varaiya, 1993), for a highway operating under normal conditions, can be found in Rao and Varaiya, (1994). The design presented there assumes a fully automated highway and uses a dynamic model of the coordination and regulation layers obtained through extensive simulations under normal operation conditions.

Lane change proportions, desired speeds and maximum platoon size are possible control variables, in that order of priority. The only control that has been fully developed and implemented in SmartPath (Eskafi *et al.*, 1992) is the lane change proportions. The control is heuristic and is derived based on four constraints:

(a) the vehicles should not miss their chosen exits;
(b) the capacity usage should be maximized;
(c) lane changes should not result in speed degradation; and
(d) shorter travel times are preferred.

These constraints are implemented through three control laws. The first one is intended to balance the traffic across the lanes. The second law specifies which cars must change lanes to reach their exit while maintaining the traffic balance. The third law acts to avoid significant increases in travel times. The speed on any section of the highway is guaranteed to be within certain bounds and the maximum value of the acceleration is also prescribed. It is demonstrated that, even with the use of simple control policies, it is possible to reduce the delays caused by incidents on the highway. The paper does not, however, consider the stability issues that arise when such control policies are implemented, and the results are difficult to generalize to arbitrary cases.

In Papageorgiou *et al.*, (1990), the parameters for a spatially discretized traffic flow model of the Southern Boulevard Perpherique of Paris are identified. The vehicles are of course under manual control. The goal here is to stabilize the traffic to a desired density and flow. The only control is on-ramp metering. As in Karaaslan *et al.*, (1990), the major problem in the absence of feedback control is that congestion and driver behavior prevent realization of the full highway capability. The authors linearize their model and apply the linear quadratic technique to determine the metering control based on density and flow information at various positions on the highway. In simulations, it is shown that with feedback control, congestion is decreased and the highway is able to sustain an otherwise unrealizable capacity. The control action at each on-ramp is mainly determined by the traffic condition local to the on-ramp. Hence, the control can be approximated by a distributed control.

The goal of this paper is to present a link layer control that regulates aggregate traffic conditions defined by density and velocity profiles to their appropriate values, while acting within the hierarchical structure of PATH. Our approach to the problem of designing a control system for the link layer is significantly different from those in the literature reviewed above. Since we are considering a fully automated AHS, we do not assume a certain *a priori* behavior of the vehicles (i.e. we do not attempt to control mixed automated and manual traffic) and we do not include coordination and regulation layer dynamics in our link layer model. An implicit assumption in this work is that the closed loop dynamics of the regulation layer control system have a sufficiently high bandwidth so that they can adequately track the reference velocity commands issued by the link layer. With regard to the coordination layer, the research presented in this paper does consider the interface dynamics between the link and coordination layer controllers. Thus, in particular, the link layer controller presented in this paper is not directly specifying platoon sizes, since platoon sizes are currently specified by the coordination layer. However, we will present these results in a future paper.

Another consequence of our approach is that the partition of the highway into different sections (spatial discretization) as well as the sampling time of the link layer control should also be determined by the bandwidth requirements of the link layer temporal and spatial dynamics. For this reason, to derive the link layer controller, we use a spatially and temporally continuous model (described by a partial differential equation) of the highway obeying only the law of conservation of vehicles.

We also make the assumptions that the dynamics associated with the velocity control of the vehicles and the change lane action are extremely fast, as compared to the dynamics of the link layer, and can be neglected. We point out that while the assumption on the dynamics of controlling velocity is realistic, the assumption on the change lane maneuvers, which currently take 3 to 6 s (Chee and Tomizuka, 1995), has to be analyzed more carefully in the future.

We assume that for each conceivable scenario (e.g. normal traffic condition, stopped vehicle on highway, blocked or closed lane), a desired traffic condition on the highway consistent with its capability under that circumstance can be prescribed. The desired traffic condition is encoded by the pair consisting of a desired density profile $K_d(x, t)$ and a desired velocity field $V_d(x)$ such that the vehicle flow rate at different positions on the highway is given by: $\phi_d(x, t) = K_d(x, t)V_d(x)$. The desired density profile determines the desired concentration of vehicles in the highway as a function of position and time, whereas the desired velocity field specifies how cars should be maneuvered so as to maintain the desired density profile and flow rate. The specific design of $K_d(x, t)$ and $V_d(x)$ depends on the demand and capability of the highway, as well as on the presence or absence of extraordinary circumstances such as accidents or lane closures. For example, under normal operating condition, $(K_d(x, t), V_d(x))$ may be found as follows: if the vehicles are traveling at velocity V, based on the capabilities of the vehicles and the highway, one can define the safety spacing $\Delta(V)$ between cars (see Frankel *et al.*, 1995). Assume that the platoon size is 1. The maximum traffic flow rate sustainable by vehicles traveling at the velocity V and at a distance $\Delta(V)$ apart is $f(V) = V/[L + \Delta(V)]$ where L is the length of the vehicle. Then given the flow rate, ϕ_d, one can determine the steady state $(K_d, V_d) = (\phi_d/V_d, V_d)$ where V_d satisfies $f(V_d) > \phi_d$. In general, the determination of the traffic condition involves some form of optimization and is work in progress.

The link layer is assumed to have a repertoire of (K_d, V_d) pairs, each of which encapsulates a strategy to deal with a particular situation on the highway. The link layer control law described below stabilizes the actual density and velocity at the desired values.

We investigate traffic stabilization control for three highway topologies:

Single lane highway. The highway consists of only one automated lane and the control is the longitudinal velocity of the traffic.

Discrete lane highway. The highway is modeled as a discrete set of automated lanes. The control in this case is the longitudinal velocities and the rates of proportion of vehicles to change lanes.

Dense lane highway. The highway consists of a two dimensional orientable manifold where the lanes are the flow lines of the desired velocity fields which densely populate the highway. The control here is the velocity field.

FLOW STABILIZATION ON A ONE LANE HIGHWAY

Consider a one lane highway which is parameterized by $x \in [0, L]$, schematically shown in Fig. 2. The evolution of the density on the highway $K(x, t)$ at any time t and position x is given by:

$$\frac{\partial}{\partial t}K(x, t) = -\frac{\partial}{\partial x}\{(K(x, t)V(x, t)\}, \tag{1}$$

where $V(x, t)$ is the traffic velocity. The control objective is to stabilize traffic by commanding a velocity profile $V(x, t)$ such that $K(x, t)$ is close to a desired density profile $K_d(x, t)$ and that the traffic moves with a desired velocity $V_d(x) > 0$. Specification of both the desired velocity and density determines the desired flow rate, $\phi_d(x, t) = K_d(x, t)V_d(x)$. The desired velocity profile $V_d(x)$ is supposed to be determined as a function of the highway capacity and it is therefore assumed to be time invariant. As recently shown (Broucke and Varaiya 1995), for a given highway topology and a set of inlet and outlet flow conditions, the velocity profile which yields an optimum highway utilization and maximum capacity is indeed time invariant. The desired density profile $K_d(x, t)$ is allowed to be time varying as long as $K_d(x, t)$ and $V_d(x)$ are consistent with eqn (1), i.e. eqn (1) is satisfied with $K(x, t) = K_d(x, t)$ and $V(x, t) = V_d(x, t)$ for all x and t.

Define the density error $\tilde{K}(x, t) := K(x, t) - K_d(x, t)$. We propose the control:

$$V(x, t) = V_d(x) + V_c(x, t) \tag{2}$$

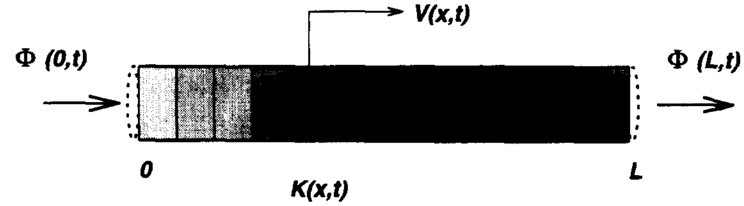


Fig. 2. One lane mass conservation model of highway.

$$V_c(x,t) = -\zeta(x,t)\frac{\partial}{\partial x}\left\{V_d(x)\tilde{K}(x,t)\right\} \tag{3}$$

where $\zeta(x,t) \geqslant 0$.

Substituting the control law into eqn (1),

$$\frac{\partial\tilde{K}(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left\{\tilde{K}(x,t)V_d(x)\right\} - \frac{\partial}{\partial x}\left\{K(x,t)V_c(x,t)\right\} \tag{4}$$

For $u:\mathcal{H}\rightarrow\mathcal{R}$ a real valued function on $\mathcal{H}=[0,\mathcal{L}]$, denote the L_2 norm by $\| u \|>_2^2 := \int_{\mathcal{H}} u^2\mathrm{d}x$. The following theorem states that with eqn (3) as the control, the desired traffic condition is stable in the L_2 sense.

Theorem A. Consider the single lane highway and a desired traffic condition $[K_d(x,t), V_d(x)]$ that is consistent with eqn (1). Suppose that the inlet flow rate is $\phi(0,t) = K_d(0,t)V_d(0)$, then under the control law (3) with $\zeta(x,t) \geqslant 0$ and $\zeta(0,t) = \zeta(L,t) = 0$, the equilibrium density error $\tilde{K}(x,t) = 0$ is L_2 stable in time; i.e. for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $\| \tilde{K}(\cdot,0) \|_2 \leqslant \delta$, then $\forall t \geqslant 0$, $\| \tilde{K}(\cdot,t) \|_2 \leqslant \varepsilon$. In fact, there exists a scalar α *s.t.* $\| \tilde{K}(\cdot,t) \|_2 \leqslant \alpha \| \tilde{K}(\cdot,0) \|_2$.

Proof: Consider the following Lyapunov functional:

$$W(t) = \frac{1}{2}\int_0^L \tilde{K}(x,t)^2 V_d(x)\mathrm{d}x.$$

Differentiating with respect to time and substituting eqn (4),

$$\begin{aligned} \dot{W}(t) &= \int_0^L \dot{\tilde{K}}(x,t)V_d(x)\mathrm{d}x \\ &= -\int_0^L \tilde{K}(x,t)V_d(x)\frac{\partial}{\partial x}\left\{\tilde{K}(x,t)V_d(x)\right\} + \tilde{K}(x,t)V_d(x)\frac{\partial}{\partial x}\{K(x,t)V_c(x,t)\}\mathrm{d}x \end{aligned} \tag{5}$$

Notice that the first term is an exact differential:

$$\frac{1}{2}\frac{\partial}{\partial x}\left\{\tilde{K}(x,t)V_d(x)\right\}^2 = \tilde{K}(x,t)V_d(x)\frac{\partial}{\partial x}\left\{\tilde{K}(x,t)V_d(x)\right\}.$$

Using the Leibnitz rule, and then substituting $V_c(x,t)$ using eqn (3), the second term in eqn (5) can be written as:

$$\begin{aligned} &\tilde{K}(x,t)V_d(x)\frac{\partial}{\partial x}\left\{K(x,t)V_c(x,t)\right\} \\ &= \frac{\partial}{\partial x}\left\{K(x,t)V_c(x,t)\tilde{K}(x,t)V_d(x)\right\} - K(x,t)V_c(x,t)\frac{\partial}{\partial x}\left\{\tilde{K}(x,t)V_d(x)\right\} \\ &= \frac{\partial}{\partial x}\left\{K(x,t)V_c(x,t)\tilde{K}(x,t)V_d(x)\right\} + \zeta(x,t)K(x,t)\left\{\frac{\partial}{\partial x}\left\{\tilde{K}(x,t)V_d(x)\right)\right\}^2. \end{aligned}$$

Hence,

$$\begin{aligned} \dot{W}(t) = &-\int_0^L \frac{\partial}{\partial x}\left\{\frac{1}{2}\left\{\tilde{K}(x,t)V_d(x)\right\}^2 + K(x,t)V_c(x,t)\tilde{K}(x,t)V_d(x)\right\} \\ &+ \zeta(x,t)K(x,t)\left\{\frac{\partial}{\partial x}\left\{\tilde{K}(x,t)V_d(x)\right\}\right\}^2\mathrm{d}x \end{aligned}$$

$$\begin{aligned} \dot{W}(t) = &-\frac{1}{2}\left\{\tilde{K}(L,t)V_d(L)\right\}^2 + \frac{1}{2}\left\{\tilde{K}(0,t)V_d(0)\right\}^2 \\ &+ \frac{1}{2}\zeta(L,t)K(L,t)\frac{\partial}{\partial x}\left\{\tilde{K}(L,t)V_d(L)\right\}^2 - \frac{1}{2}\zeta(0,t)K(0,t)\frac{\partial}{\partial x}\left\{\tilde{K}(0,t)V_d(0)\right\}^2 \\ &- \int_0^L \zeta(x,t)K(x,t)\left\{\frac{\partial}{\partial x}\left\{\tilde{K}(x,t)V_d(x)\right\}^2\right\}^2\mathrm{d}x \end{aligned} \tag{6}$$

Because $\zeta(0, t) = \zeta(L, t) = 0$, the third and fourth terms in eqn (6) vanish. Under the assumption that the inlet condition $\phi(0, t) = V_d(0)K_d(0, t)$, and the fact that $V(0, t) = V_d(0)$, $K(0, t) = K_d(0, t)$ and $\tilde{K}(0, t) = 0$ for all t. Hence,

$$\dot{W}(t) \leqslant \frac{1}{2}\left[\tilde{K}(0, t)V_d(0)\right]^2 - \frac{1}{2}\int_0^L \zeta(x, t)K(x, t)\left\{\frac{\partial}{\partial x}\{\tilde{K}(x, t)V_d(x)\}\right\}^2 \mathrm{d}x$$

$$= -\frac{1}{2}\int_0^L \zeta(x, t)K(x, t)\left\{\frac{\partial}{\partial x}\{\tilde{K}(x, t)V_d(x)\}\right\}^2 \mathrm{d}x \leqslant 0.$$

The last inequality is obtained by noting that the density $K(x, t)$ whose dynamics are given by eqn (1) must remain positive and the constraint on the gain $\zeta(x, t) \geqslant 0$. Thus, $W(t) \leqslant W(0)$. If we define $\underline{V_d} = \inf_{x\in\mathcal{H}} V_d(x)$, and $\overline{V_d} = \sup_{x\in\mathcal{H}} V_d(x)$, then,

$$\underline{V_d} \parallel \tilde{K}(\cdot, t) \parallel_2^2 \leqslant W(t) \leqslant W(0) \leqslant \overline{V_d} \parallel \tilde{K}(\cdot, 0) \parallel_2^2 .$$

Thus, for all $t \geqslant\ = 0$, $\parallel \tilde{K}(\cdot, t) \parallel_2 \leqslant \alpha \parallel \tilde{K}(\cdot, 0) \parallel_2$ for $\alpha = \sqrt{\overline{V_d}/\underline{V_d}}$. L_2 stability follows.

Remark

To understand this control law, consider the simplified case when both V_d and K_d are constants. The control becomes:

$$V(x, t) = V_d - \zeta(x, t)V_d\frac{\partial\tilde{K}(x)}{\partial x}$$

So if the density is higher downstream than it is upstream, $\frac{\partial\tilde{K}(x)}{\partial x}(x, t) > 0$ and the control law decreases the velocity. This has the effect of preventing a pile up downstream. This can be interpreted as a density-homogenizing control law.

Notice that the control law eqn (3) is distributed, i.e. velocity command at position x is determined by the density nearby.

The control law is very simple: it involves taking the gradient of the weighted density error. The weighting factor in this case is $V_d(x)$ which is given by the desired traffic condition.

FLOW STABILIZATION ON A DISCRETE-LANE HIGHWAY

Consider now a highway of length L, consisting of n lanes (in Fig. 3, $n = 3$). A parameterization is the pair (x, i) with $x \in [0, L]$, and $1 \leqslant i \leqslant n$. Vehicles on the highway can change to adjacent lanes. In addition to the ability to control the velocity in the longitudinal direction on each lane, we also assume that we can control the proportions of cars that switch lanes. One advantage of controlling the proportions as opposed to the absolute number of cars that change lane is that the density on the highway can be guaranteed to be non-negative.

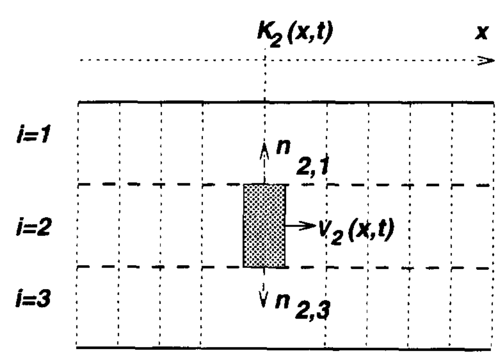


Fig. 3. Discrete lane highway model: control is by specifying the longitudinal velocity and the proportion of lane changes from one lane to another.

Desired traffic condition

The desired traffic condition on the discrete lane highway is specified by the following:

1. $\overline{V}_{di} : [0, L] \to \mathcal{R}^+$, a strictly positive desired time invariant velocity profile for each lane $i = 1, \cdots, n$;
2. $\overline{K}_{di} : [0, L] \times \mathcal{R} \to \mathcal{R}^+$, the desired possibly time varying desired density profile for each lane $i = 1, \cdots, n$;
3. $\overline{n}^d_{i,j} : [0, L] \to \mathcal{R}$, a rate of change lane proportion profile for each pair of adjacent lanes, $1 \leqslant i, j \leqslant n$, and $| i - j | = 1$. The meaning of the change lane density profile will become clear when the dynamics of the traffic density are discussed.
4. $\sigma : [0, L] \times \{1, \ldots, n\} \to [0, L]$, a connectivity function that satisfies:
   - For each $i \in [1, n], \sigma_i(\cdot) := \sigma_i(\cdot, i)$ is monotone increasing and onto; (hence $\sigma(\cdot)$ is invertible).
   - When a car changes from lane i to lane j, and the current position of the car is (x, i) then it is located at (y, j) after the lane change, where

$$y = \sigma_j^{-1}[\sigma_i(x)]. \tag{7}$$

Hence, the connectivity function defines the location of the vehicles during an instantaneous change lane maneuver.

Remarks

- Taking the inverse for eqn (7) shows that if vehicles at (x, i) can switch lanes to (y, j), then vehicles from (y, j) that switch to lane i must reappear at (x, i).
- The pre-image of σ of the intervals in any partition $\{[L_0, L_1], [L_1, L_2], \cdots, [L_{m-1}, L_m]\}$ of $[0, L]$ where $0 = L_0 < L_1 \ldots < L_m = L$ disjointly covers $\mathcal{H}$, i.e.:

$$\mathcal{H} = \biguplus_{1 \leqslant k \leqslant m} \sigma^{-1}([L_{k-1}, L_k]) + \bigcup_{1 \leqslant i \leqslant n} (L, i).$$

The connectivity σ allows one to define a new longitudinal coordinate for the highway so that the lane change between lane i and lane j takes place in the same new longitudinal coordinate. The coordinate transformation for each lane $\rho : [0, L] \times \{1, \cdots, n\} \to [0, L] \times \{1, \cdots, n\}$ is given by (see Fig. 4):

$$\rho : (x, i) \to (\sigma(x, i), i). \tag{8}$$

One can think of $\mathcal{H}$ as a *bundle* over the base space $[0, L]$, with $\sigma : \mathcal{H} \to [0, L]$ being the projection. For each $s \in [0, L]$, the inverse image $\sigma^{-1}(s)$ is the fiber above s.

For technical reasons, some restrictions on σ, $\overline{n}^d_{i,j}$ and $\overline{V}_{di}$ are made.

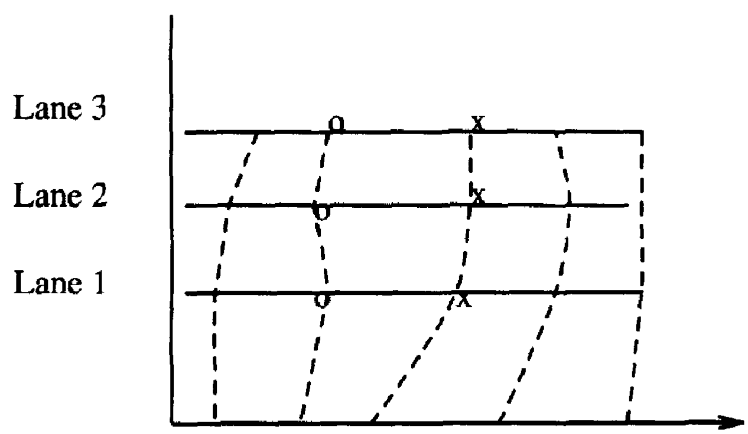


Fig. 4. The connectivity function σ induces a change of longitudinal coordinates of the highway. All the x and o on the three lane highway are mapped to the same longitudinal coordinate s_1 and s_2. Transfer of vehicles in lane change maneuvers takes place among fibers shown here as the dotted lines.

Assumption A. There exists $\mathcal{P}$, a (finite) m part partition of the base space $[0, L]$,

$$\mathcal{P} = \{[L_0, L_1), [L_1, L_2), \cdots, [L_{m-1}, L_m]\}$$

where $0 = L_0 \leqslant L_1 \cdots \leqslant L_m = L$ so that within each odd interval, $[L_{2(k-1)}, L_{2k-1}), k = 1 \cdots p$ the desired lane change proportions $\bar{n}^d_{i,j}(\sigma_i^{-1}(s)) = 0, s \in [L_{2(k-1)}, L_{2k-1}), \forall i, j \leqslant n$; and on each even interval, $[L_{2k-1}, L_{2k}), \bar{n}^d_{i,j}(\sigma_i^-(s))$ may not vanish, for $s \in [L_{2k-1}, L_{2k})$. If at the beginning of the highway $\bar{n}^d_{i,j}(\sigma_i^-(0)) \neq 0$ for some $i, j \in [1 \cdots, n]$, then we can take the first interval to have zero length, i.e. $L_0 = L_1 = 0$.

Assume also that a scalar function r_c defined on the even intervals (where the desired lane change is non-zero):

$$r_c(s) \bigcup_{k|2k \leqslant m} [L_{2k-1}, L_{2k}) \to \mathcal{R}^+,$$

exists such that for all $i, j \leqslant n$

$$r_c(s) := \frac{\partial \sigma i}{\partial x}(xi(s))\bar{V}_{di}(x_i(s)) = \frac{\partial \sigma j}{\partial x}(x_j(s))\bar{V}_{dj}(x_j(s))\bar{V}_{dj}(x_j(s)) \tag{9}$$

where $x_j(s) = \sigma_j^{-1}(s)$.

This assumption imposes a condition on the relation between the geometry of lane change maneuver (i.e. the connectivity $\sigma(\cdot,\cdot)$ and the desired velocity profile $V_d(\cdot)$ in the even intervals of the partitions where the desired lane change may not be zero. The situation is illustrated in Fig. 5

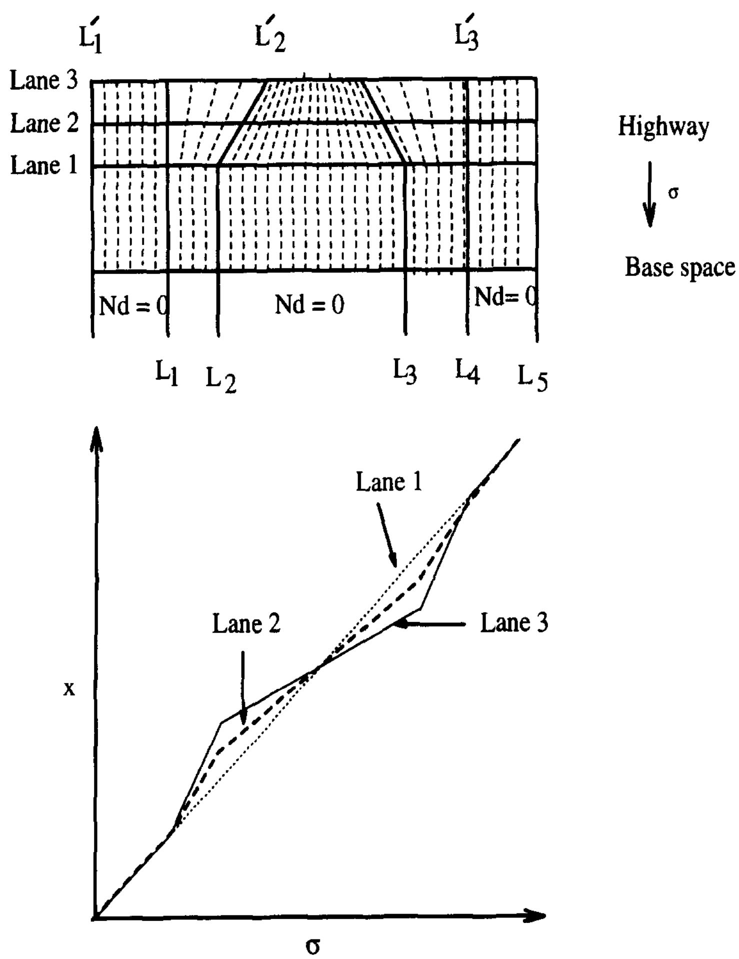


Fig. 5. A four-lane highway where the $\bar{V}_{d1} = 1$, $\bar{V}_{d2} = 3/2$, and $V_{d3} = 2$, $\bar{n}^d_{i,j} = 0$ on the intervals $[0, L_1)$, $[L_2, L_3)$ and $[L_4, L_5]$. On the intervals $[L_1, L_2)$ and $[L_3, L_4)$, eqn (9) is satisfied by setting $\frac{\partial \sigma_i}{\partial x}(\sigma^{-1}(s)) = 1/V_{di}(s)$.

for a three lane highway where the $\overline{V}_{d1} = 1, \overline{V}_{d2} = 3/2$, and $\overline{V}_{d3} = 2$. $\overline{n}^d_{i,j} \neq 0$ in the second and fourth regions. In these regions, eqn (9) is satisfied by requiring the fibers to be more spread out in lanes with higher desired velocities.

Remarks

- $\sigma(\cdot,\cdot)$ specifies how the lane changes connect the different lanes. In practice, given $\overline{V}_d(x)$ and $\overline{n}^d_{i,j}(x)$, the partition $\mathcal{P}$ will be chosen and σ determined via eqn (9) in intervals of $\mathcal{P}$ where the desired lane change is non-zero. These can take place off-line.
- In normal situations, one would favor lane changes that take place 'vertically', i.e. the vehicles changing lane are located at the same longitudinal position (in the original coordinate x). Equation (9) then says that within an interval of $\mathcal{P}$ where the desired lane changes are possible, the desired velocities on these lanes must be the same (see Fig. 6).
- 'Vertical' lane changes between lanes i and j, i.e. $\sigma(x,i) = \sigma(x,j)$, are also possible if the desired traffic condition does not involve lane changes.
- 'Non-vertical' lane change maneuver, i.e. lane changes that involve different longitudinal positions, can be realized by a 'vertical' lane change followed by speeding up (or slowing down) for a short period of time to gain (or lose) a specified distance.

Density dynamics on $\mathcal{H}$

Let $\overline{K}_i(x,t)$ be the density (number of vehicles/unit length) on lane i at position x and time t, and $\overline{V}_i(x,t)$ be the velocity profile for lane i which is under control of the link layer controller.

Using the notation $y_{i,j}(x) := \sigma_j^{-}[\sigma_j(x)]$, the dynamics of the density on $\mathcal{H}$ are given by:

$$\frac{\delta \overline{K}_i}{\delta t}(x,t) = \begin{cases} -\frac{\partial \overline{K}_i(x,t)\overline{V}_i(x)}{\partial x} & -\overline{K}_i(x,t)(\overline{n}_{i,i-1}(x,t) + \overline{n}_{i,i+1}(x,t)) \quad , i = 2,\cdots,n-1 \\ & +\overline{K}_{i-1}(y_{i-1,i}(x))\overline{n}_{i-1,i}(y_{i-1,i}(x),t) \\ & +\overline{K}_{i+1}(y_{i+1,i}(x),t)\overline{n}_{i+1,i}(y_{i+1,i}(x),t) \\ -\frac{\partial \overline{K}_i(x,t)\overline{V}_i(x)}{\partial x} & -\overline{K}_i(x,t)\overline{n}_{i,i+1}(x,t) \quad , i = 1 \\ & +\overline{K}_{i+1}(y_{i+1,i}(x),t)\overline{n}_{i+1,i}(y_{i+1,i}(x),t) \\ -\frac{\partial \overline{K}_i(x,t)\overline{V}_i(x)}{\partial x} & -\overline{K}_i(x,t)\overline{n}_{i,i-1}(x,t) \quad , i = n \\ & +\overline{K}_{i-1}(y_{i-1,i}(x),t)\overline{n}_{i-1,i}(y_{i-1,i}(x),t) \end{cases} \tag{10}$$

where $\overline{n}_{i,j}(x,t)$ is the proportion of cars/unit time that have to change lane from lane i at position x to lane j at time t. Notice that in a change lane maneuver, cars appearing at position (x,i) from lane j will come from location $(y_{j,i}(x), j), y_{j,i}, (x) = \sigma_j^{-1}(\sigma_i(x))$. Hence, the right hand side of eqn (10) consists of three types of terms which account for:

- traffic flow along the lane,
- traffic leaving the lane, and
- traffic entering the lane.

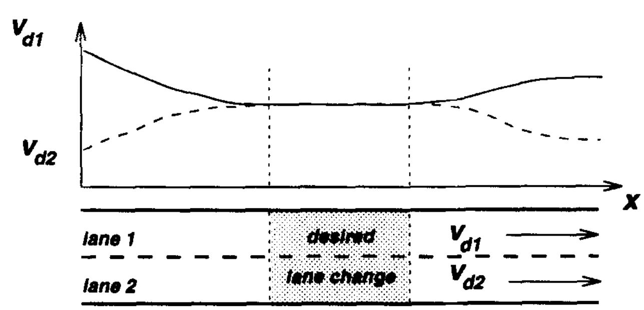


Fig. 6. When changing lane at the same longitudinal position, the desired velocity fields on the lanes involved in lane change are the same.

Notice also that $\bar{n}_{i,j}(x, t)$ determines the proportion of cars to switch from lane i to lane j in unit time as it is multiplied by $\bar{K}_i(x, t)$ in eqn (10).

Let $\bar{V}_{di}(x)$ and $\bar{K}_{di}(x, t)$ be the desired velocity field and the density field of lane i, and $\bar{n}^d_{i,j}(x)$ the desired lane change proportion such that they satisfy the discrete lane traffic dynamics eqn (10) with $\bar{V}_i(x), \bar{K}_i(x, t)$ and $\bar{n}_{i,j}(x)$ replaced by $\bar{V}_{di}(x), \bar{K}_{di}(x, t)$ and ${}^d_{i,j}(x)$.

We suppose that $\bar{V}_i(x, t)$ and $\bar{n}_{i,j}(x, t)$ are the control variables. Similar to the control for the single lane highway model, we propose control laws consisting of terms that are determined by the desired traffic conditions and terms determined by the current traffic condition:

$$\bar{V}_i(x, t) = \bar{V}_{di}(x) + \bar{V}_{ci}(x, t) \tag{11}$$

$$\bar{n}_{i,j}(x, t) = \bar{n}^d_{i,j}(x) + \bar{n}^c_{i,j}(x, t). \tag{12}$$

Using the coordinate transformation ρ in eqn (8) induced by the connectivity σ, we now rewrite the dynamics eqn (10) in a more compact matrix form.

Notation. To express the dynamics in the σ induced coordinates, we define,

$$K_i(s, t) := \bar{K}_i(\sigma_i^{-1}(s), t) \qquad K_{di}(s, t) := \bar{K}_{di}(\sigma_i^{-1}(s), t)) \tag{13}$$

$$V_i(s, t) := \bar{V}_i(\sigma_i^{-1}(s), t) \qquad V_{di}(s) := (\bar{V}_{di}(\sigma_i^{-1}(s)) \tag{14}$$

$$n_{i,j}(s, t) := \bar{n}_{i,j}(\sigma_i^{-1}(s), t) \qquad \bar{n}^d_{i,j}(s, t) := \bar{n}^d_{i,j}(\sigma_i^{-1}(s), t) \tag{15}$$

$$V_{ci}(s, t) := \bar{V}_{ci}(\sigma_i^{-1}(s), t) \qquad n^c_{i,j}(s, t) := \bar{n}^c_{i,j}(\sigma_i^{-1}(s), t) \tag{16}$$

We shall use the variable s exclusively to denote the longitudinal position in the new coordinates.

Denote also by the vectors,

$$\begin{aligned}
\mathbf{K}(s, t) &: = [K_1(s, t), K_2(s, t), \cdots, K_n(s, t)]^T, \\
\tilde{\mathbf{K}}(s, t) &: = [K_1(s, t) - K_{d1}(s, t), K_2(s, t) - K_{d2}(s, t), \cdots, K_n(s, t) - K_{dn}(s, t)]^T,
\end{aligned}$$

the density and the density error; and by the following matrices, the total desired velocity, feedback velocity action, desired lane change proportion and the feedback lane change proportion:

$$\mathbf{V}_d(s) := \begin{pmatrix} V_{d1}(s) & 0 & \cdots \\ 0 & V_{d2}(s) & 0\cdots \\ & \ddots & \\ 0 & \cdots 0 & V_{dn}(s) \end{pmatrix}, \qquad \mathbf{V}_c(s) := \begin{pmatrix} V_{c1}(s) & 0 & \cdots \\ 0 & V_{c2}(s) & 0\cdots \\ & \ddots & \\ 0 & \cdots 0 & V_{cn}(s) \end{pmatrix}$$

$$\mathbf{N}_d(s) := \begin{pmatrix} -n^d_{1,2}(s) & n^d_{2,1}(s) & 0 & \cdots \\ n^d_{1,2}(s) & -n^d_{2,1}(s) - n^d_{2,3}(s) & n^d_{3,2}(s) & \cdots 0 \\ 0 & n^d_{2,3}(s) & -n^d_{3,2}(s) - n^d_{3,4}(s) & \cdots 0 \\ & \vdots & & \\ 0 & \cdots & n^d_{n-1,n}(s) & -n^d_{n,n-1}(s) \end{pmatrix} \tag{17}$$

$$\mathbf{N}_c(s) := \begin{pmatrix} -n^c_{1,2}(s) & n^c_{2,1}(s) & 0 & \cdots \\ n^c_{1,2}(s) & -n^c_{2,1}(s) - n^c_{2,3}(s) & n^c_{3,2}(s) & \cdots 0 \\ 0 & n^c_{2,3}(s) & -n^c_{3,2}(s) - n^c_{3,4}(s) & \cdots 0 \\ & \vdots & & \\ 0 & \cdots & n^c_{n-1,n}(s) & -n^c_{n,n-1}(s) \end{pmatrix} \tag{18}$$

Furthermore, denote,

$$\Sigma(s) = \begin{pmatrix} \frac{\partial\sigma_1}{\partial x}(\sigma_1^{-1}(s)), & 0 & \cdots & \\ 0 & \frac{\partial\sigma_2}{\partial x}(\sigma_2^{-1}(s)) & 0\cdots & \\ & \ddots & & \\ 0 & \cdots 0 & 0 & \frac{\partial\sigma_n}{\partial x}(\sigma_n^{-1}(s)) \end{pmatrix} \tag{19}$$

Notice that when lane change action is 'vertical', the connectivity $\sigma(x, i) = x$, so that $\sum(s) = \mathbf{I}$, the identity matrix.

In these notations, the dynamics of the density error $\tilde{\mathbf{K}}K(s, t)$ is given by:

$$\begin{aligned} \frac{\partial\tilde{\mathbf{K}}}{\partial t}(s, t) = &- \Sigma(s)\frac{\partial}{\partial s}\left\{\mathbf{V}_d(s)\tilde{\mathbf{K}}(s, t)\right\} + \mathbf{N_d}(s)\tilde{\mathbf{K}}(s, t) \\ &- \Sigma(s)\frac{\partial}{\partial s}\{\mathbf{V}_c(s, t)\} + \mathbf{N}_c(s, t)\mathbf{K}(s, t). \end{aligned} \tag{20}$$

Control law

Using the matrix notations above, we propose control laws for the feedback component of the longitudinal control $\mathbf{V}_c(s, t)$ and the elements in the change lane proportion $\mathbf{N}_c(s, t)$ in eqns (11) and (12).

First we define two matrix functions which transform the coordinates of the density error $\tilde{\mathbf{K}}(s)$. The first matrix function $\mathbf{A} : [0, L] \rightarrow \mathcal{R}^{n\times n}$ is always invertible and combines the densities of the lanes into another set of densities (the densities of a set of fictitious lanes), whose dynamics are decoupled. The second matrix function, $\mathbf{P} : [0, L] \rightarrow \mathcal{R}^{n\times n}$, is a diagonal matrix that assigns weights to the decoupled densities.

Matrix $\mathbf{A}(s)$. The matrix $\mathbf{A}(s) \in \mathcal{R}^{n\times n}$ is computed as follows.

Let $\mathcal{P} = \{[L_0, L_1], [L_1, L_2), .., [L_{m-1}, L_m]\}$ be a partition of $\mathcal{H}$ with $L_0 = 0$ and $L_m = L$ be the partition in *Assumption A*. Recall p is the number of odd intervals and in these intervals $n^d_{i,j} = 0$. We shall define $\mathbf{A}(s)$ by solving a linear matrix differential equation on intervals $[L_{2(k-1)}, L_{2k})$ for $k = 1, \ldots, p-1$, and $[L_{2(p-1)}, L]$. The values of $\mathbf{A}(s)$ will be reset appropriately at the beginning of these intervals. Specifically;

Step 0. Set $k = 1$.

Step 1. Set $\mathbf{A}(L_{2(k-1)}) = p_k\mathbf{I}$ where

$$p_k = \begin{cases} 1 & \textit{if } k = 1 \\ \frac{r_c[L_{2(k-1)}]\underline{\sigma}\left(\mathbf{A}(L^-_{2(k-1)})\right)}{\underline{\sigma}\left(\Sigma(L_{2(k-1)})\mathbf{V}_d(L_{2(k-1)})\right)} & \textit{if } k \neq 1 \end{cases} \tag{21}$$

where $\mathbf{A}(s_0^-)$ denotes the limit of $\mathbf{A}(\cdot)$ from the left at s_0^-, i.e. $\mathbf{A}(s_0^-) = \lim_{s\rightarrow s_0}\mathbf{A}(s)$ with s increasing; $\underline{\alpha}(\mathbf{X})$ and $\bar{\sigma}(\mathbf{X})$ denote the minimum and maximum singular values of the matrix $\mathbf{X}$, respectively, $r_c(\cdot)$ is the scalar function in eqn (9).

Step 2. Solve the following n^2 ordinary differential equation over the interval $s \in [L_{2(k-1)}, L_{2k})$ if $k < p$ and over $s \in [L_{2(p-1)}, L]$ if $k = p$:

$$\frac{d}{ds}\mathbf{A}(s) = -\mathbf{A}(s)\left\{\mathbf{N}_d(s)\mathbf{V}_d^{-1}(s)\Sigma^{-1}(s) + \frac{\partial\sum}{\partial s}(s)\Sigma^{-1}(s)\right\} \tag{22}$$

Step 3. Set $k = k + 1$

Step 4. Repeat steps 1 through 3 until $k = p + 1$.

Matrix $\mathbf{P}(s)$. The matrix $\mathbf{P}(s)$ is defined to be:

$$\mathbf{P}(s) = \Sigma(s)\mathbf{V}_d(s) \tag{23}$$

Proposition $\mathbf{P}$(s). can be written as

$$\mathbf{P}(s) = \mathbf{A}(s)\Sigma(s)\mathbf{V}_d(s)\mathbf{A}^{-1}(s). \tag{24}$$

Proof: First notice that eqn (22) is a set of linear ODE, so $\mathbf{A}(s) = \mathbf{A}(L_{2(k-1)})\Phi(s, L_{2(k-1)})$ where $\Phi[s, L_{2(k-1]})$ is the transition matrix. Because transition matrices are invertible, and $\mathbf{A}(L_{2(k-1)})$ is set to be invertible in eqn (21) for each k, $\mathbf{A}^{-1}(s)$ exists for $s \in [0, L]$.

Each of the intervals $[L_{2(k-1)}, L_{2k})$ can be written as $[L_{2(k-1)}, L_{2k-1}) \cup [L_{2k-1}, L_{2k})$ such that the $n^d_{i,j}(s) = 0$ on $[L_{2(k-1)}, L_{2k-1})$. On the first portion, the right hand side of eqn (22) is diagonal, so $\mathbf{A}(s)$ is diagonal on $[L_{2(k-1)}, L_{2k-1})$ since $\mathbf{A}(L_{2(k-1)})$ is also diagonal by eqn (21). In this case, each component on the right hand side of eqn (24) is diagonal and commutes with each other. Hence $\mathbf{A}(s)\Sigma(s)\mathbf{V}_d(s)\mathbf{A}^{-1}(s) = \mathbf{A}(s)\mathbf{A}^{-1}(s)\sum(s)\mathbf{V}_d(s)\mathbf{P}(s)$.

On $[L_{2k-1}, L_{2k})$, by eqn (9) in assumption A, $\Sigma(s)\mathbf{V}_d(s) = r_c(s)\mathbf{I}$. Hence, the r.h.s. of eqn (24) can be written as:

$$\begin{aligned}\mathbf{A}(s)\Sigma(s)\mathbf{V}_d(s)\mathbf{A}^{-1}(s) &= \mathbf{A}(s)(r_c(s)\mathbf{I})\mathbf{A}^{-1}(s)\\ &= r_c(s)\mathbf{I} = \Sigma(s)\mathbf{V}_d(s).\end{aligned}$$

We now define the transformed density error $\mathbf{F}(s, t) \in \mathcal{R}^n$:

$$\mathbf{F}(s, t) = \mathbf{A}^T(s)\mathbf{P}(s)\mathbf{A}(s)\tilde{\mathbf{K}}(s, t) \tag{25}$$

The control laws are given by:

$$\mathbf{V}_c(s, t) = -\zeta(s, t)diag\left\{\frac{\partial}{\partial s}[\Sigma(s)\mathbf{F}(s, t)]\right\} \tag{26}$$

$$n^c_{i,j}(s, t) \geqslant 0, \text{ if } F_i(s, t) - F_j(s, t) > 0 \tag{27}$$

where $\zeta(s, t) \geqslant 0$ is a gain function, and is set to zero at the inlets and outlets of the highway, and at the end of an interval where the desired change lane proportion is non-zero.

Remark

$\mathbf{A}(s)$ and $\mathbf{P}(s)$ are computed off-line once the desired traffic condition $(\mathbf{V}_d, \mathbf{K}_d, \mathbf{N}_d)$ and the connectivity σ are given.

Under the above control law, the discrete lane highway has the following property.

Theorem B. Consider the n–discrete lane highway model. Suppose that the desired velocity profile $\overline{V}_{di}$, the desired change lane proportion profile $\overline{n}^d_{i,j}$ and the desired density profile $\overline{K}_{di}$ satisfy the highway traffic dynamics, eqn (10) and that the connectivity assumption A is satisfied. For the density error for each lane i, $\tilde{K}_i(s, t) = K_i(s, t) - \tilde{K}_{di}(s, t)$, where $K_i(s, t)$ and $K_{di}(s, t)$ are related to $\overline{K}_i(x, t)$ and $\tilde{K}_{di}(x, t)$ by eqn (13). Denote the L_2 norm to be:

$$\| \tilde{\mathbf{K}}(\cdot, t)^2_2 \| = \int_0^L \sum_{i=1}^n \tilde{K}_i(s, t)^2 \mathrm{d}s.$$

Assume that the inlet condition at each lane i is such that the flow rate $\phi_i(0, t) = \overline{K}_{di}(0, t)\overline{V}_{di}(0)$, then under the control law, eqn (25–27), there exists an $0 < \alpha < \infty$, such that,

$$\| \tilde{\mathbf{K}}(\cdot, t) \|_2 \leqslant = \alpha \| \tilde{\mathbf{K}}(\cdot, 0) \|_2$$

Hence, under this control law, the equilibrium $\tilde{K}_i(s) = 0, \forall s \in [0, L]$ is L_2 stable.

Proof. See appendix A.

Remarks

- The connectivity function $\sigma(x, i)$ and $r_c(s)$ in eqn (9) and the coordinate transformation $\mathbf{A}(s)$ in eqn (25) are necessary to calculate the transformed density error $\mathbf{F}(s, t)$. However, once the desired traffic profile is chosen (i.e. $\mathbf{V}_d$ and $\mathbf{K}_d$) these functions can be computed off-line in a straight forward way (by solving an initial value problem for a linear ODE). Thus, the real time computation necessary is limited to taking simple directional gradients for a weighted density error function.
- The control laws eqns (26) and (3) are similar in that for the longitudinal control, the feedback term $\mathbf{V}_c$ is given by a partial derivative of a weighted error $V_d(x)\tilde{K}(x, t)$ in the single lane case, and $\Sigma(s)\mathbf{F}(s, t)$ in the discrete lane case. It is easy to see that $\Sigma(s)\mathbf{F}(s, t)$ reduces to $V_d(x)\tilde{K}(x, t)$ when $n = 1$.
- The feedback control for lane change $\mathbf{N}_c$ is done by comparing the error between the $F_i(s)$ and $F_j(s)$. We can interpret this control as the gradient of $\mathbf{F}$ in the transverse direction.
- Similar to the single lane case, control for the discrete lane highway can be distributed: it requires only traffic information near the particular longitudinal displacement along the highway (in the s coordinate).
- Desired lane change proportions are usually non-zero near incidents or exits. In most other regions on the highway, desired lane change is generally zero. It is for this reason that we partition the highway in assumption 3.1. Notice that in regions where the desired lane change proportion is zero, one can set $\Sigma(x) = x$ and $\mathbf{A}(s)$ becomes a multiple of the identity matrix.

Simulation results

To illustrate the effectiveness of the control law described in this section, we show simulation results of a three lane automated AHS. The initial condition of the highway is one in which all three lanes have uniform and equal density along the highway. Moreover, vehicles in all lanes are moving at the same constant velocity. At the beginning of the simulation a new desired density, velocity and lane change flow profile ($\mathbf{K}_d(x)$, $\mathbf{V}_d(x)$, $\mathbf{N}_d(s)$) is imposed to the link layer controller. This desired flow profile requires all vehicles in lanes 1 and 3 to change lane to lane 2. Vehicles in lane 1 are requested to change lane in the middle region of the freeway, while vehicles in lane 3 are requested to change lane at 3/4 of the length of the highway. The desired velocity for all three lanes before, during and after the lane changes is constant and equal to the initial condition. This situation is schematically depicted in Fig. 7.

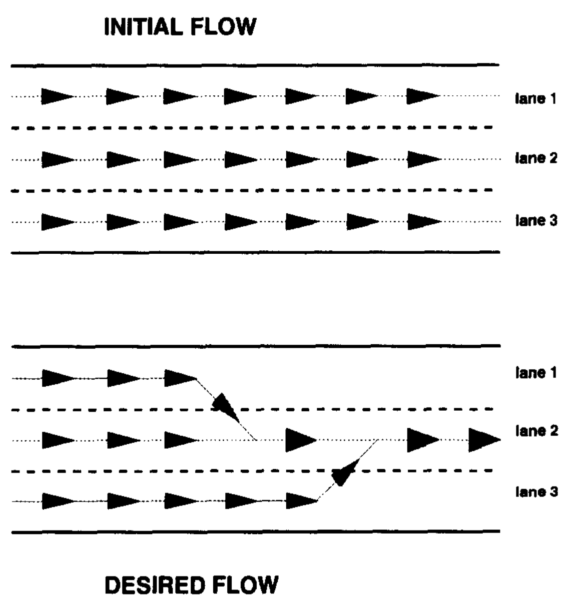


Fig. 7. Initial and desired velocity field profile.

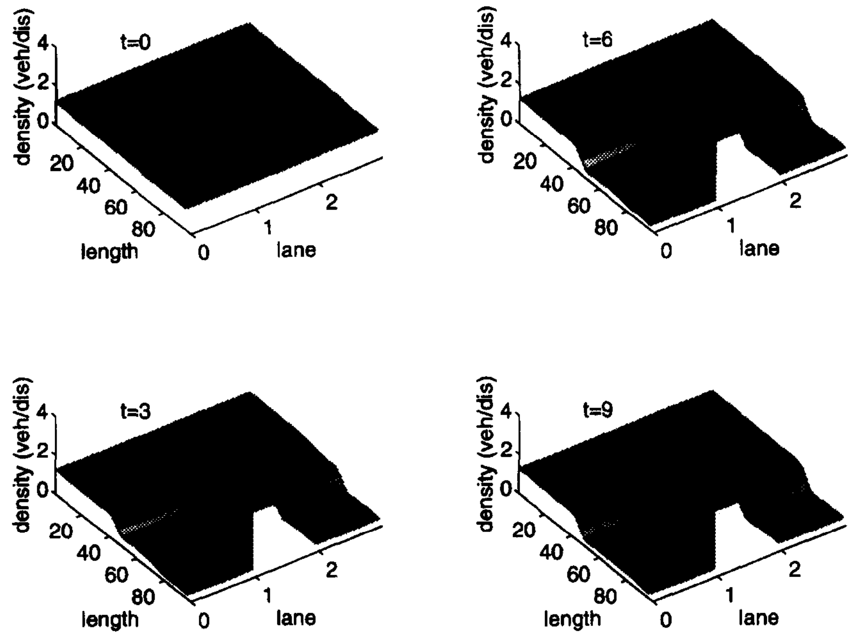


Fig. 8. Link layer controller response with feedforward and feedback control.

Under these conditions, the control given by eqns (11)–(12) and (22)–(26) simplifies to

$$\mathbf{V}(x, t) = \mathbf{V}_d(x) + \mathbf{V}_c(x, t) \tag{28}$$

$$\mathbf{N}(x, t) = \mathbf{N}_d(x) + \mathbf{N}_c(x, t) \tag{29}$$

where

$$\mathbf{V}_c(x, t) = -\zeta_1 \frac{\partial}{\partial x}\left(\mathbf{V}_d(x)\tilde{\mathbf{K}}(x, t)\right) \tag{30}$$

$$n_{i,j}^c(x, t) \geqslant 0, \text{ if } \tilde{K}_i(x, t) - \tilde{K}_j(x, t) > 0. \tag{31}$$

Figure 8 shows the response of the closed loop feedback system under the control given by eqns (28)–(30) and

$$n_{i,j}^c(x, t) = \zeta_2(\tilde{K}_i(x, t) - \tilde{K}_j(x, t)), \text{ if } \tilde{K}_i(x, t) - \tilde{K}_j(x, t)) > 0, \tag{32}$$

where ζ_1 and ζ_2 are two positive constants.

Notice that the convergence from initial to final flow pattern in Fig. 8 is very fast. The convergence rate of the closed loop system can be adjusted by changing the feedback gains ζ_1 and ζ_2.

It should be mentioned that, in this example, both the initial and desired flow patterns satisfy the inlet vehicle flow conditions. In fact, the system will converge to the desired profile even when only the desired velocity and lane change commands are used in the control law (i.e. the constants ζ_1 and ζ_2 are set to zero). However, as Figs 9 and 10 illustrate, the convergence of the closed loop system is much faster when the feedback law is used.

DENSE LANE HIGHWAY

We now turn to a two dimensional highway model (Fig. 11). Let the highway $\mathcal{H}$ be a two dimensional orientable manifold. From the principle of vehicle conservation, the dynamics of the traffic density satisfy the following partial differential equation:

$$\frac{\partial K(x, t)}{\partial t} = -\nabla \cdot \phi(x, t); \qquad \phi(x, t) = K(x, t)V(x, t) \tag{33}$$

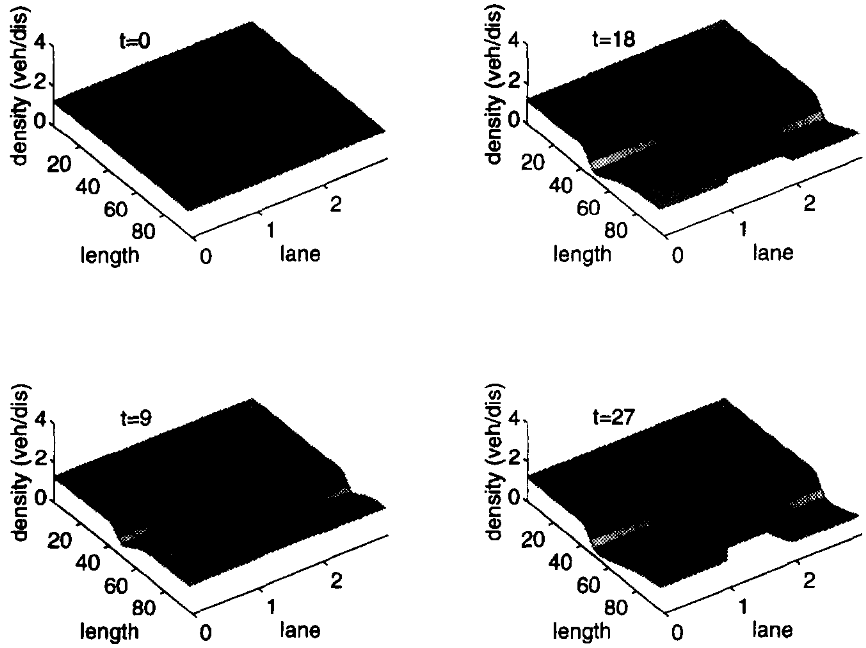


Fig. 9. Link layer controller response with only feedforward control.

where $K(x, t)$ is the density at position x and time t, $\nabla\cdot$ represents the divergence operator in the normal Cartesian coordinates. We assume that $V(x, t)$ is a velocity field of the traffic flow that we can command and that $K(x, t)$ is available for measurement.

Assume that the desired traffic condition is given by a pair of density and velocity fields $(K_d(x, t), V_d(x))$ where $K_d(\cdot, t) : \mathcal{H} \to \mathcal{R}$, and $V_d(\cdot) : \mathcal{H} \to T\mathcal{H}$ and $V_d(x) \neq 0$ for all $x \in \mathcal{H}$. Moreover, $K_d(x, t)$ and $V_d(x)$ satisfy eqn (33).

Let $\psi : \mathcal{H} \times \mathcal{R} \to \mathcal{H}$ be the flow of V_d, i.e. ψ satisfies:

$$\frac{\partial \psi}{\partial t}\Big|_{x,t=0} = V_d(x), \qquad \psi(x, 0) = x.$$

The flowlines of the velocity field cover the highway. We assume every flow line either:

1. Forms a closed circuit, i.e. $\exists T_x$ s.t. $\psi(x, T_x) = x$; or
2. if the flowline starts on the boundary of the highway, it will eventually exit.

These flowlines play the roles of lanes in this model. Because the flow lines densely pack the highway, we call this 2-D highway model a dense lane highway model. We further assume that there exists a one dimensional submanifold $\mathcal{S}$ of $\mathcal{H}$ that intersects each flow line at one point.

Under these assumptions, there exists a diffeomorphism,

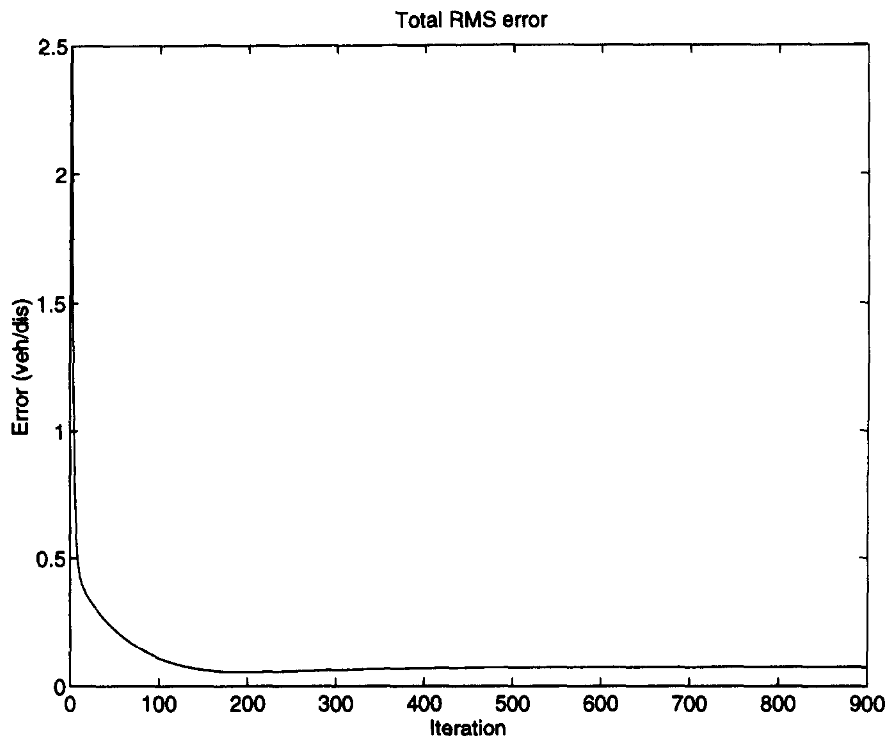
$$\rho : \mathcal{H} \to \mathcal{R} \times \mathcal{S}.$$

We can interpret $\mathcal{S}$ to be a section of the highway (usually the entrance). The assumptions on the flow lines (lanes) implies that for each $h \in \mathcal{S}$, $\exists T_h > 0$ which is the time for the flow originating for $h \in \mathcal{S}$ either to exit the highway or to return to h (Fig. 11). Thus, $x \in \mathcal{H}$ can be relabeled by a pair given by $x \mapsto (\sigma, h) \in \mathcal{R} \times \mathcal{S}$.

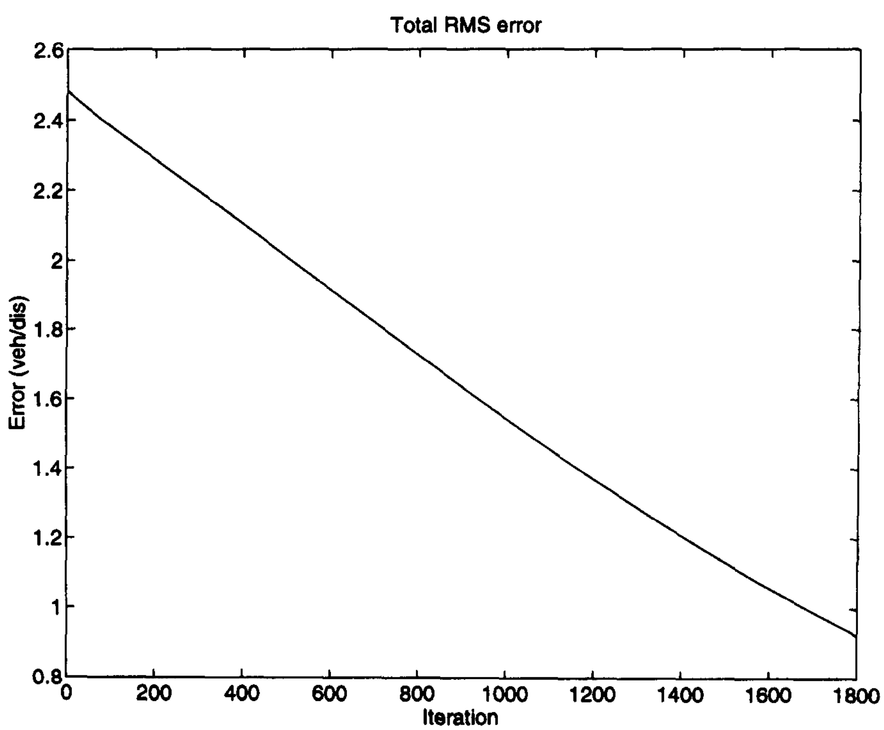
Remarks

- Under the new parameterization $x \mapsto (\sigma, h)$, h labels which lane x is on, and σ describes how far along x is on the lane. In fact, σ measures the time a particle takes to reach the point (σ, h) from the 1-d submanifold S if it travels along the flow line determined by h at the speed specified by the velocity field V_d.

- This coordinate transformation is only a function of the desired traffic condition, and can be computed (if necessary) *off-line*. It is normally not necessary to compute the coordinate transformation in order to derive the control law.



with feedback and feedforward



with feedforward only

Fig. 10. Maximum density error as a function of time.

Density error dynamics

The dynamics of the density function $K : \mathcal{H} \times \mathcal{R}^+ \to \mathcal{R}$ in eqn (33) can be rewritten in the form:

$$\frac{\partial K(x, t)}{\partial t} = -div_\mu(K(x, t)V(x, t)), \tag{34}$$

where μ is the standard Cartesian volume 2-form $dx \wedge dy$, and $div_\mu \chi$ denotes the divergence of the vector field χ with respect to the volume form μ, and is defined as (see Abraham *et al.*, 1988 for details):

$$\mathcal{L}\chi\mu = div_\mu \chi_\mu,$$

and $V(x, t)$ is the control velocity field.

The control goal is to stabilize the density field $K(x, t)$ to the desired density field $K_d(x, t)$, where $Kd(x, t)$ and $Vd(x)$ satisfy:

$$\frac{\partial K_d(x, t)}{\partial t} = -div_\mu(K_d(x, t)V_d(x)). \tag{35}$$

Although we require that $V_d(x)$ is time invariant, $K_d(x, t)$ can be time varying.

The control law we propose is of the form:

$$V(x, t) = V_d(x) + V_c(x, t).$$

Let $\tilde{K}(x, t) = K(x, t) - K_d(x, t)$ be the density error. The dynamics of the density error are given by:

$$\frac{\partial \tilde{K}(x, t)}{\partial t} = -div_\mu\big(\tilde{K}(x, t)V_d(x)\big) - div_\mu(K(x, t)V_c(x)). \tag{36}$$

Flow stabilization

The coordinates $x \mapsto (\sigma, h)$ induce a new volume form defined (locally) by,

$$\mu_1 := d\sigma \wedge dh.$$

This form has the property that $div_{\mu 1} V_d \mu_1 = 0$. To see this, first notice that in the (σ, h) coordinates, the desired velocity field takes the simple form $V_d = \frac{\partial}{\partial \sigma}$. Then, expand the definition of *div*, and apply the Cartan's magic formula (54) to obtain,

$$div_{\mu 1} V_d \mu_1 = \mathcal{L} V_d \mu_1 = dV_d \rfloor \mu_1 + V_d \rfloor d\mu_1$$

where $\chi \rfloor \alpha$ denotes the contraction of the p-form α by the vector field χ,

$$(\chi \rfloor \alpha)(\chi_2, \chi_3, \cdots, \chi p) := \alpha(\chi, \chi_2, \chi_3, \cdots, \chi p).$$

Since $d\mu_1 = 0$ (d of any volume form is 0), $V_d \rfloor \mu_1 = dh$ and $ddh = 0$, $div_{\mu 1} V_d = 0$.

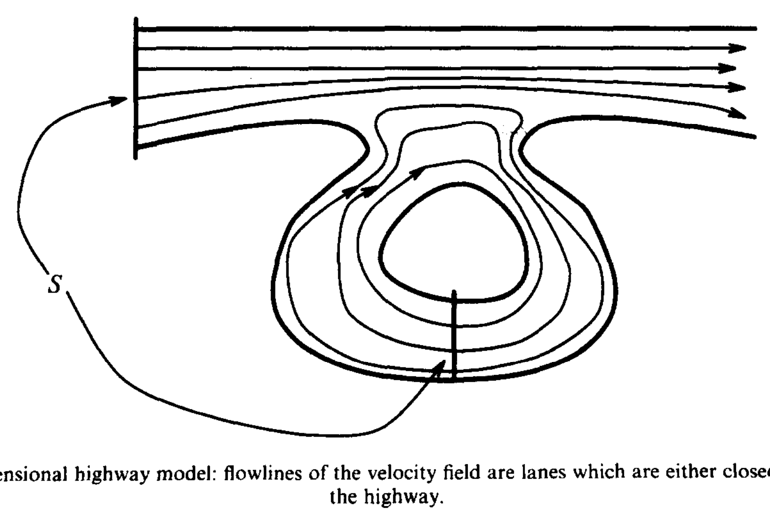


Fig. 11. Two dimensional highway model: flowlines of the velocity field are lanes which are either closed, or enter and exit the highway.

The new volume form μ_1 can be computed intrinsically by imposing the condition,

$$div_{\mu 1} V_d \mu_1 = 0.$$

Since any 2-volume forms on orientable manifolds are related by a function, $\exists p : \mathcal{H} \to \mathcal{R}$,

$$\mu = p \cdot \mu_1. \tag{37}$$

So computing p is equivalent to computing μ_1. The function $p : \mathcal{H} \to \mathcal{R}$ will be used in the control law.

The function $p : \mathcal{H} \to \mathcal{R}^+$ is computed as follows: using the divergence formula (see (B5) in Appendix B),

$$div_\mu V_d = div_{\mu 1} V_d + \frac{1}{p} \mathcal{L} V_d p.$$

By imposing the condition that,

$$div \mu_1 V d \mu_1 := \mathcal{L} V_d \mu_1 = 0, \tag{38}$$

we obtain,

$$div_\mu V_d = \frac{1}{p} \mathcal{L} V_d p = \mathcal{L} V_d \ln p. \tag{39}$$

where ln is the natural logarithm. Equation (39) defines an ordinary differential equation on each flow line. Let $\mathcal{S}$ be the 1-dimensional submanifold that generates $\mathcal{H}$. Specify $p(x)$ on $\mathcal{S}$ to be positive, integrate eqn (39) along the flow lines of V_d to obtain $p(x)$ on $\mathcal{H}$. Notice that in this computation, if p is set to be positive on $\mathcal{S}$, p will remain positive.

We are now ready to define the control law:

$$V(x, t) = V_d(x) + V_c(x, t) \tag{40}$$

where V_c satisfies:

$$\begin{gathered} \mathbf{d}(p\tilde{K}) V_c = \mathcal{L}_{Vc} p \tilde{K} \leqslant 0, \\ V_c(x, t) \text{ is parallel to } \partial H \text{ whenever } x \in \partial \mathcal{H}. \end{gathered} \tag{41}$$

One possible definition of V_c is:

$$V_c(x, t) = -\zeta(x, t) \text{grad}(\tilde{K}(x, t) p(x)) \tag{42}$$

where the gradient is taken with respect to to any Riemannian metric (possibly time varying) on $\mathcal{H}$, and $\zeta(x, t) \geqslant 0$ is a scalar gain function which vanishes on $\partial\mathcal{H}$.

Theorem C. For the two dimensional highway model, suppose that the conditions on the desired velocity field and the desired density field are satisfied. Assume also that the flow $\phi(x, t) = K_d(x, t) V_d(x)$ for $x \in$ *inlet* $\subset \partial\mathcal{H}$ (where the flow lines enter the highway). Define the L_2 norm to be $\| K(\cdot, t) \|_2^2 = \int_{\mathcal{H}} \tilde{K}(x, t)^2 \mu$ (where μ is the standard Cartesian volume form $dx_1 \wedge dx_2$).

Under the control law, (40) with (41) satisfied, the equilibrium $\tilde{K}(x, t) = 0, \forall x \in \mathcal{H}$, is Lyapunov stable in the L_2 sense, i.e. for every $\varepsilon > 0, \exists \delta > 0$, such that if $\| \tilde{K}(\cdot, 0) \|_2 < \delta$, then $\| \tilde{K}(\cdot, t) \|_2 > \varepsilon$ for all $t \geqslant 0$.

Proof: See appendix B.

Remarks

- As in the case for the single lane and discrete lane highway models, the control law (26) is distributed, i.e. requires only local density information.
- The control (26) consists of a feedforward term $V_d(x)$ which is determined by the desired traffic condition and a feedback term $V_c(x)$ which is the gradient of a weighted density error.
- The weighting in this case is given by the function $p(x)$ which can be computed off-line via (39). Equation (39) is an initial value problem.

Simulation

To demonstrate the effectiveness of the control law, a simulation is performed. We simulated the controller on a circular highway and the desired density profile is a stationary hole: a region of low density in the middle of the highway (Figs 12 and 13). The traffic that leaves the highway on the right hand side of the figure re-enters on the left. The desired velocity field is shown in Fig. 14.

The initial density profile is shown in Fig. 15. Notice that the density profile after 25 time units, shown in Fig. 16, is nearly exactly the same as the desired density profile. The time function of the maximum error is shown in Fig. 17. It should be noted that in this example, the actual density profile will not converge to the desired profile without feedback control.

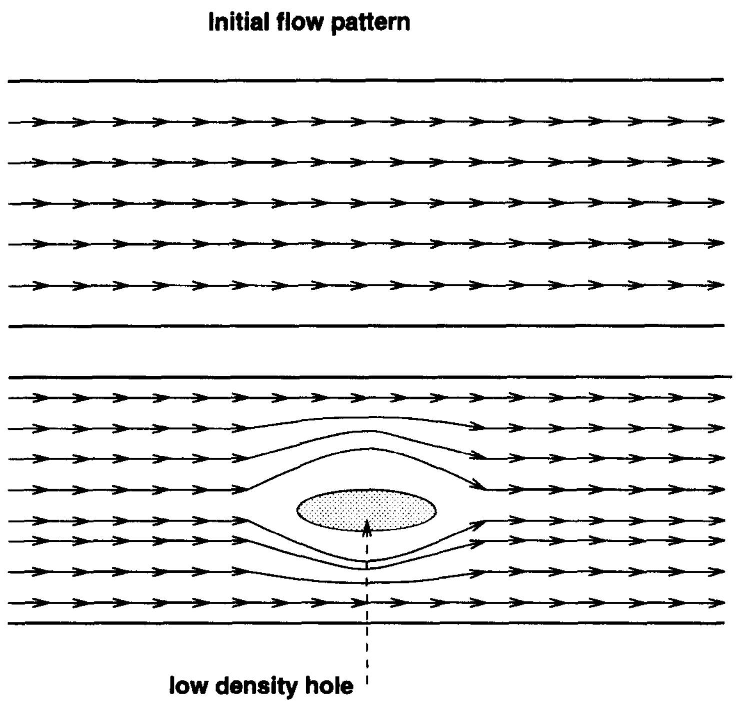


Fig. 12. Simulation example: the desired traffic condition is a density profile that conforms to a 'stationary hole' on a circular highway.

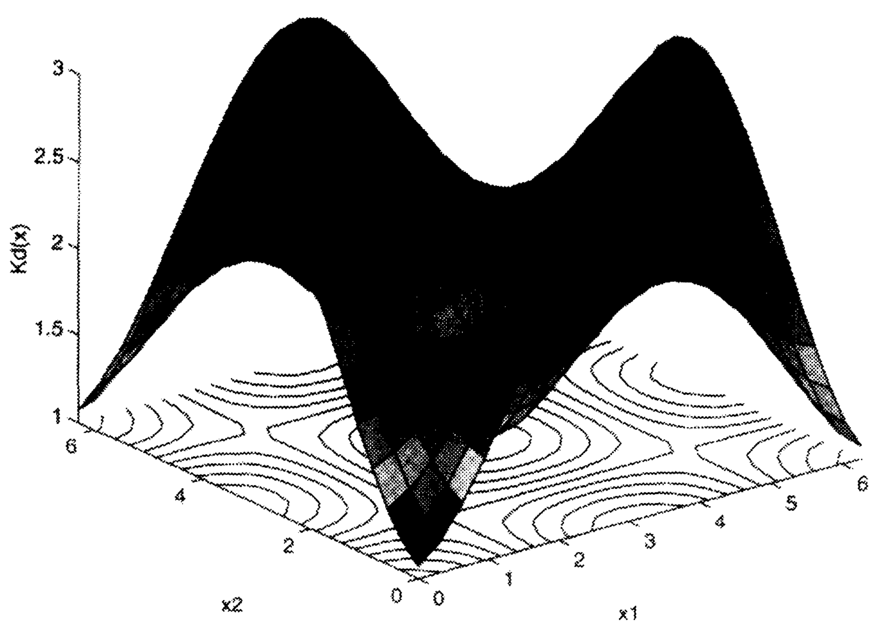


Fig. 13. Time invariant desired density field $K_d(x, t)$ on the same circular highway.

CONCLUSIONS

Research on link layer control for AHS system is discussed. Control laws are proposed for the stabilization of traffic conditions (specified by velocity and density profiles). Controls are considered for three highway model topologies: a single lane highway, a multiple discrete lane highway, and a two dimensional highway with arbitrary traffic flow pattern. The control law in each case is simple, consisting of a feedforward action and feedback action obtained by a generalized gradient. The link layer controllers designed have the following characteristics:

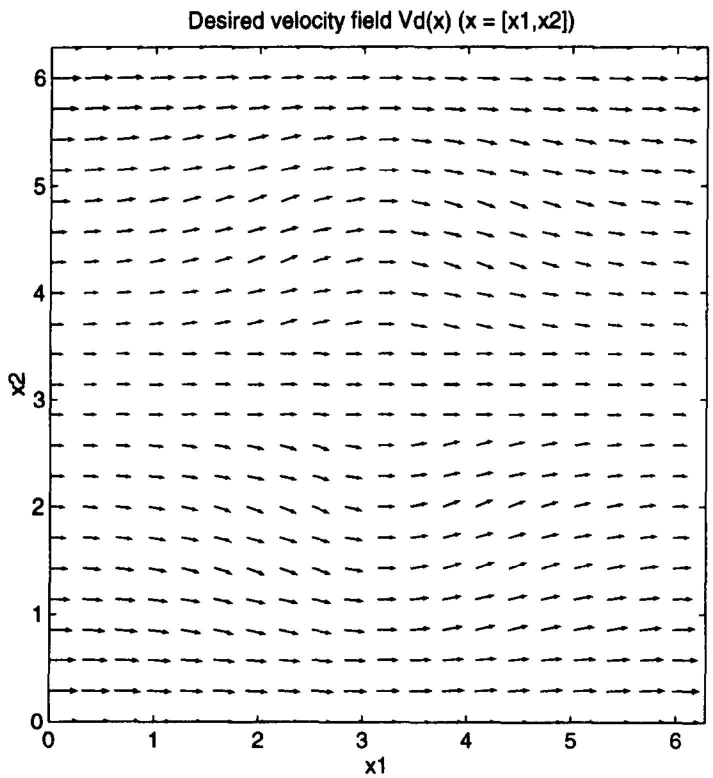


Fig. 14. Quiver diagram showing the desired velocity field $V_d(x)$ on a circular highway parameterized by $x \in [0, 2\pi] \times [0, 2\pi]$, with x being the circular coordinate. Direction and length of arrows indicate the direction and magnitude of the velocity field.

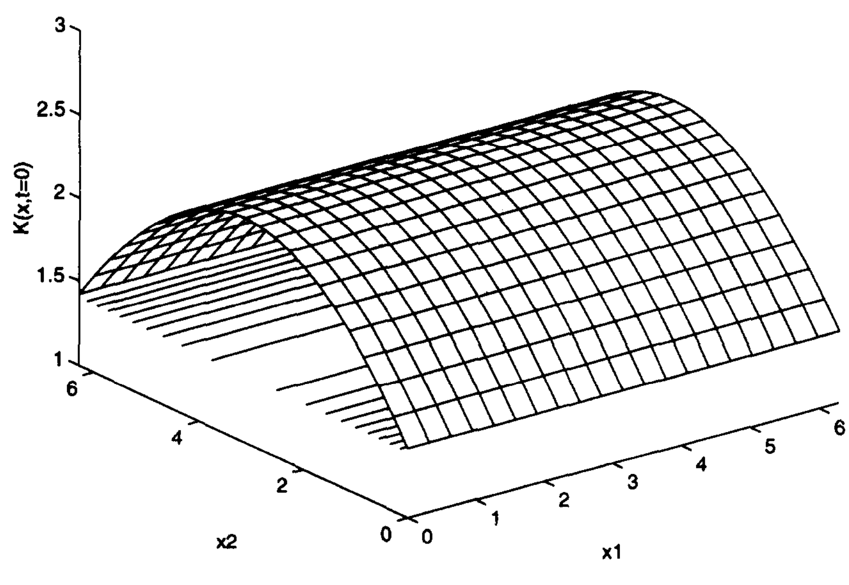


Fig. 15. Initial ($t = 0$) density field $K(x, 0)$ on the highway.

Decentralized control scheme

The calculations needed to determine the value of the control at each section of the highway are based only on information locally available to a link layer in that section. A decentralized control scheme is suited for AHS hierarchical architectures. Our proposed control involves calculating the generalized gradient of a weighted density error. The density information may be estimated by each platoon leader based on the platoon sizes of itself and the platoons ahead and behind; as well as the distances separating the platoons (assuming backward sensors or communications with trailing platoon are available). The roadside computer will provide the weighting function and the desired traffic condition necessary to compute the weighted density error.

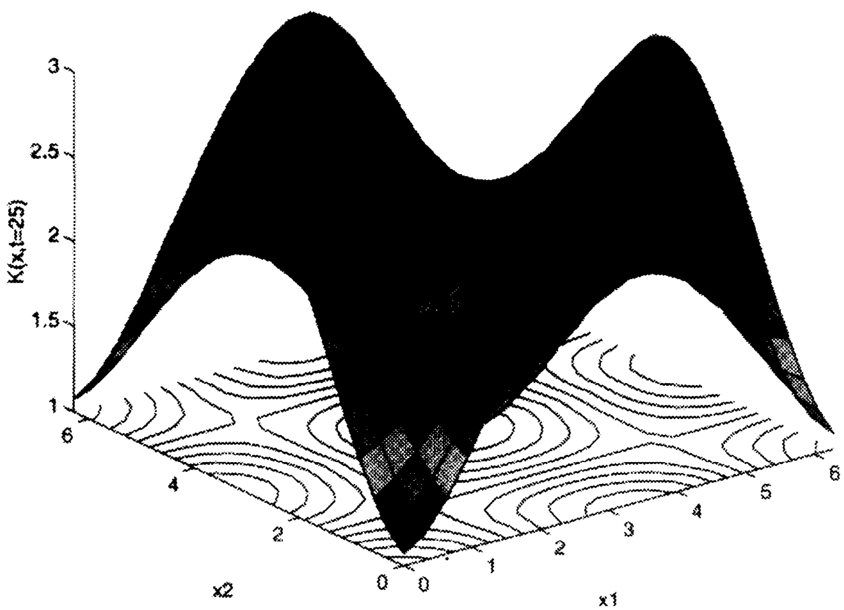


Fig. 16. Final density profile $K(x, 25)$ on the same manifold at $t = 25$ time units.

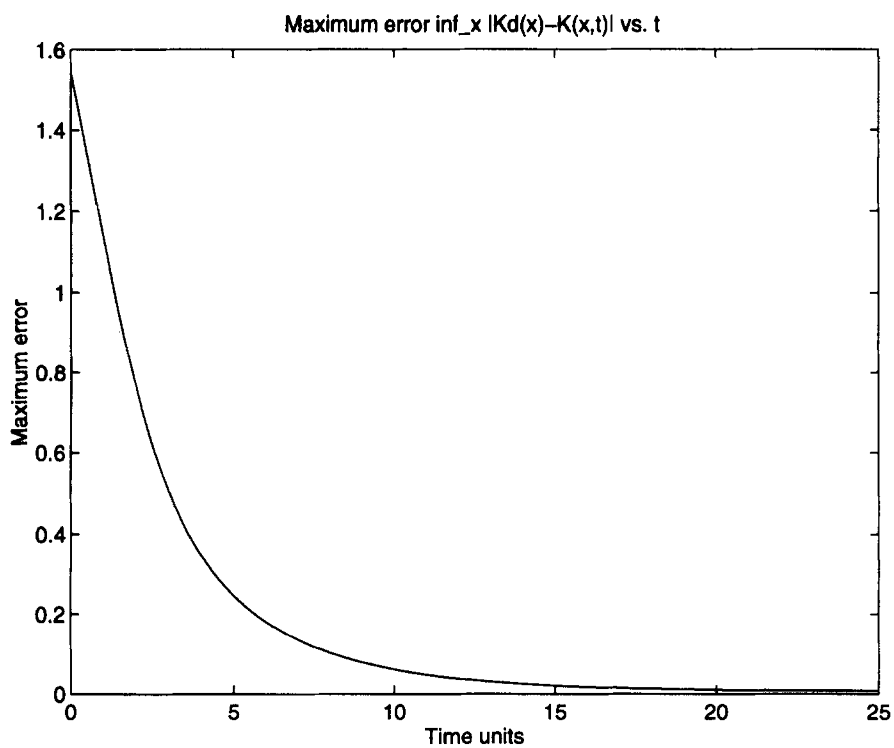


Fig. 17. Maximum error $\sup_x \mid K_d(x) - K(x, t) \mid$ as a function of time units.

Guaranteed stability

The derivation of the link layer control laws proposed in this paper was based on stability considerations. Moreover, we exploited unique properties of mass conservation laws for highway systems which allowed us to obtain remarkably simple laws within a rigorous framework. Examples of these properties were the facts that highway density always remains non-negative and that exact differentials can easily be derived for the topologies considered. An examination of the closed loop dynamics reveals that the terms resulting from the feedback laws proposed in this paper have a similar structure to the diffusion equation terms in heat transfer problems. These terms are always dissipative and stabilizing. In addition, the control system will not become unstable even when the desired flow pattern is incompatible with the inlet and outlet boundary conditions. In this case, the controller attempts to decrease the error density L_2 norm by uniformly distributing the error throughout the highway.

Ability to handle degraded modes of operation:

If the degraded conditions of operation, currently under development for the PATH AHS architecture (Lygeros *et al.*, 1995) are coded as desired flow patterns, the link layer controllers here presented can also be used to stabilize the traffic. Simulation of such a case as presented in the section Simulation Results.

Acknowledgements—This research is part of a larger effort to develop control systems for AHS in degraded mode operations under project MOU-135. Besides the authors, this group consists of Professor Shankar Sastry, Professor Pravin Varaiya, Dr Akash Deshpande, Dr Datta Godbole, Tony Lindsey, Dr John Lygeros, Anuj Puri, Ekta Singh from the U.C.B. EECS Department and Mireille Broucke from PATH. Their input to this work is gratefully acknowledged.

REFERENCES

Abraham, R., Marsden, J. E. and Ratiu, T. (1988) *Manifolds, Tensor Analysis, and Applications.* Springer-Verlag, New York.

Broucke, M. and Varaiya, P. (1995) A theory of traffic flow in automated highway systems. *Transportation Research C*, **4**(4), 181–210.

Chee, W. and Tomizuka, M. (1995) Lane change maneuver of automobiles for the Intelligent Vehicle and Highway Systems IVHS. *Proceedings of the 1994 American Control Conference*, pp. 3586–3589.

Chien, C. C., Zhang, Y. and Stotsky, A. (1993) Traffic density tracker for automated highway systems. *Proceedings of the 33rd IEEE CDC.*

Eskafi, F., Khorramabadi, D. and Varaiya, P. (1992) Smartpath: an automated highway system simulator. *Technical Report 92-3, of the Institute of Transportation Studies.* University of California, Berkeley.

Frankel, J., Alvarez, L., Horowitz, R. and Li, P. (1995) Safety oriented control for AVHS maneuvers. *Proceedings of the 1995 American Control Conference.* Also in UCB PATH Technical Note 94-9, 1994, and to appear in *Journal of Vehicle System Dynamics.*

Godbole, D. and Lygeros, J. (1993) Longitudinal control of the lead car of a platoon. *Technical Report 93-7 of the Institute of Transportation Studies.* University of California, Berkeley. Also in *IEEE Transactions on Vehicular Technology* Vol. 43. No. 4 pp. 1125–1135.

Godbole, D., Lygeros, J. and Sastry, S. (1994) Hierarchical hybrid control: A case study. *Proceedings of the 34th IEEE CDC.*

Karaaslan, U., Varaiya, P. and Walrand, J. (1990) Two proposals to improve freeway traffic flow. *Technical Report 90-6 of the Institute of Transportation Studies.*PATH Research Report, University of California, Berkeley.

Lygeros, J., Godbole, D. N. and Broucke, M. E. (1995) Design of an extended architecture for degraded modes of operation of IVHS. *Proceedings of the 1995 American Control Conference.*

Papageorgiou, M., Blosseville, J.-M. and Hadi-Salem, H. (1990) Modelling and real-time control of traffic flow on the southern part of boulevard peripherique in Paris: Part i: Modelling and Part ii: Coordinated on-ramp metering. *Transportation Research A*, **24**(5), 345–370.

Rao, B. S. Y. and Varaiya, P. (1994) Roadside intelligence for flow control in an intelligent vehicle and highway system. *Transportation Research C*, **2**(1), 49–92.

Varaiya, P. (1993) Smart cars on smart road: Problems of control. *IEEE Transaction on Automatic Control* **32**(2), 195–207.

APPENDIX A

Proof of Theorem B

We define a set of fictitious densities,

$$\mathbf{G}(s,t) := \mathbf{A}(s)\tilde{\mathbf{K}}(s,t), \tag{A1}$$

where $\mathbf{A}(s)$ is defined in eqns (21)–(22).

Step 1.: We claim that $\mathbf{G}(s,t)$ in eqn (A1) satisfies the following on $[0, L]$ except possibly at $\{L_{2(k-1)} \mid k = 1, \cdots, p\}$:

$$\frac{\partial}{\partial t}\mathbf{G}(s,t) = -\frac{\partial}{\partial s}\{\mathbf{P}(s)\mathbf{G}(s,t)\} - \mathbf{A}(s)\Sigma(s)\frac{\partial}{\partial s}(\mathbf{V}_c(s)\mathbf{K}(s,t)) + \mathbf{A}(s)\mathbf{N}c(s)\mathbf{K}(s,t). \tag{A2}$$

Differentiating eqn (A1) with respect to t, substituting the error dynamics (20) into the result, and then applying Leibnitz rule in reverse, we obtain,

$$\begin{aligned}\frac{\partial}{\partial t}\mathbf{G}(s,t) = &-\frac{\partial}{\partial s}\left\{\mathbf{A}(s)\Sigma(s)\mathbf{V}_d(s)\tilde{\mathbf{K}}(s,t)\right\}\\ &+\left\{\frac{\partial}{\partial s}\{\mathbf{A}(s)\Sigma(s)\}\mathbf{V}_d(s) + \mathbf{A}(s)\mathbf{N}_d(s)\right\}\tilde{\mathbf{K}}(s)\\ &- \mathbf{A}(s)\Sigma(s)\frac{\partial}{\partial s}(\mathbf{V}_c(s)\mathbf{K}(s,t)) + \mathbf{A}(s)\mathbf{N}_c(s)\mathbf{K}(s,t).\end{aligned} \tag{A3}$$

Expanding the expression in {} in the second line of eqn (A3),

$$\begin{aligned}&\frac{\partial}{\partial s}\{\mathbf{A}(s)\Sigma(s)\}\mathbf{V}_d(s) + \mathbf{A}(s)\mathbf{N}_d(s)\\ &=\frac{\partial}{\partial s}\mathbf{A}(s)\Sigma(s)\mathbf{V}_d(s) + \mathbf{A}(s)\frac{\partial}{\partial s}\Sigma(s)\mathbf{V}_d(s) + \mathbf{A}(s)\mathbf{N}_d(s)\\ &=\left[\frac{\partial}{\partial s}\mathbf{A}(s) + \mathbf{A}(s)\left(\frac{\partial}{\partial s}\Sigma(s)\Sigma^{-1}(s) + \mathbf{N}_d(s)\mathbf{V}_d^{-1}(s)\Sigma^{-1}(s)\right)\right]\Sigma(s)\mathbf{V}_d(s)\\ &=0.\end{aligned}$$

since $\mathbf{A}(s)$ satisfies eqn (22).

Substituting $\mathbf{P}(s)$ in eqn (24) into eqn (A3), we obtain eqn (A2).

Step 2. For each $k = 1, \cdots, p$, consider the Lyapunov functional on $[L_{sk}, L_{fk}]$:

$$W_k(t) = \frac{1}{2}\int_{L_{sk}}^{L_{fk}^-} \mathbf{G}^T(s,t)\mathbf{P}(s,t)\mathbf{G}(s,t)\mathrm{d}s.$$

where $L_{sk} = L_{2(k-1)}$; and $L_{fk} = L_{2k}$ if $k < p$ and $L_{fk} = L$ if $k = p$.

Its time derivative is given by:

$$\begin{aligned}\dot{W}_k(t) = &-\int_{L_{sk}}^{L_{fk}} \mathbf{G}^T(s,t)\mathbf{P}(s)\frac{\partial}{\partial s}\{\mathbf{P}(s)\mathbf{G}(s,t)\}\mathrm{d}s\\ &-\int_{L_{sk}}^{L_{fk}} \mathbf{G}^T(s,t)\mathbf{P}(s)\mathbf{A}(s)\Sigma(s)\frac{\partial}{\partial s}(\mathbf{V}_c(s,t)\mathbf{K}(s,t)) - \mathbf{G}^T(s,t)\mathbf{P}(s)\mathbf{A}(s)\mathbf{N}_c(s,t)\mathbf{K}(s,t)\mathrm{d}s\end{aligned}$$

Applying the Leibnitz rule in reverse, and using the fact that $\mathbf{P}(s)$ is diagonal,

$$\dot{W}_k(t) = -\frac{1}{2}\int_{L_{sk}}^{L_{fk}^-}\frac{\partial}{\partial s}\left\{\mathbf{G}^T(s,t)\mathbf{P}^2(s)\mathbf{G}(s,t)\right\}\mathrm{d}s \tag{A4}$$

$$-\int_{L_{sk}}^{L_{fk}^-}\frac{\partial}{\partial s}\left\{\mathbf{G}^T(s,t)\mathbf{P}(s)\mathbf{A}(s)\sum(s)\mathbf{V}_c(s,t)\mathbf{K}(s,t)\right\} \tag{A5}$$

$$-\frac{\partial}{\partial s}\left(\mathbf{G}^T(s,t)\mathbf{P}(s)\mathbf{A}(s)\sum(s)\right)\mathbf{V}_c(s,t)\mathbf{K}(s,t) \tag{A6}$$

$$+\mathbf{G}^T(s,t)\mathbf{P}(s)\mathbf{A}(s)\mathbf{N}_c(s,t)\mathbf{K}(s,t)\mathrm{d}s. \tag{A7}$$

Notice that the first two terms are exact integrals. So by the fundamental theorem of calculus,

$$\dot{W}(t) = -\frac{1}{2}\mathbf{G}^T(s,t)\mathbf{P}^2(s)\mathbf{G}(s,t)\,\Big|_{s=L_{sk}}^{s=L_{fk}^-} - \mathbf{G}^T(s,t)\mathbf{P}(s)\mathbf{A}(s)\Sigma(s)\mathbf{V}_c(s,t)\mathbf{K}(s,t)\,\Big|_{s=L_{sk}}^{s=L_{fk}^-} \tag{A8}$$

$$+\int_{L_{sk}}^{L_{fk}^-}\frac{\partial}{\partial s}\big(\mathbf{G}^T(s,t)\mathbf{P}(s)\mathbf{A}(s)\Sigma(s)\big)\mathbf{V}_c(s,t)\mathbf{K}(s,t)\mathrm{d}s \tag{A9}$$

$$-\int_{L_{sk}}^{L_{fk}^-}\mathbf{G}^T(s,t)\mathbf{P}(s)\mathbf{A}(s)\mathbf{N}_c(s,t)\mathbf{K}(s,t)\mathrm{d}s. \tag{A10}$$

Because the control law dictates that $V_c(L_{sk}, t) = V_c(L_{fk}^-, t = 0$, the second set of boundary terms vanish.

Substituting $\mathbf{F}(s,t) = \mathbf{A}^T(s)\mathbf{P}(s)\mathbf{A}(s)\tilde{\mathbf{K}}K(s,t)$ and the control law (26), the term on the third line of eqn (A10) becomes:

$$-\int_{L_{sk}}^{L_{fk}^-}\zeta(s,t)\left\{\frac{\partial}{\partial s}\big(\mathbf{F}^T(s,t)\Sigma(s)\big)\right\}diag(\mathbf{K}(s,t))\left\{\frac{\partial}{\partial s}(\Sigma(s)\mathbf{F}(s,t))\right\}\mathrm{d}s$$
$$= -\sum_{i=1}^{n}\int_{L_{sk}}^{L_{fk}^-}\zeta(s,t)\left\{\frac{\partial}{\partial s}(F_i(s,t)\Sigma(i))^2K_i(s,t)\right\}\mathrm{d}s \leqslant 0$$

since the density $\mathbf{K}(s,t) \geqslant 0$ for all s and t.

To see that the fourth term,

$$-\int_{L_{sk}}^{L_{fk}^-}\mathbf{F}^T(s,t)\mathbf{N}_c(s,t)\mathbf{K}(s,t)\mathrm{d}s \leqslant 0,$$

write out the integrand in terms of $n_{i,j}^c(s,t)$ using the structure of $N_c(s,t)$ in eqn (18). It is,

$$-\sum_{\substack{|i-j|=1\\ 1\leqslant i,j,\leqslant n}} K_i(s,t)n_{i,j}^c(s,t)\big(F_j(s,t) - F_i(s,t)\big).$$

Hence, under the control law (27) that $n_{i,j}^c(s,t) \geqslant 0$ if $F_j(s,t) - F_i(s,t) > 0$, we see that this term is indeed non-positive.

Thus, for $k = 1,\cdots,p$.

$$\dot{W}_i(t) \leqslant -\frac{1}{2}\mathbf{G}^T(L_{fk}^-,t)\mathbf{P}^2(L_{fk}^-)\mathrm{G}(L_{fk}^-,t) + \frac{1}{2}\mathbf{G}^T(L_{sk},t)\mathbf{P}^2(L_{sk})\mathbf{G}(L_{sk},t).$$

Step 3. Now consider the Lyapunov function for the whole highway:

$$W(t) = \sum_{i=1}^{p} W_k(t)$$

Notice that,

$$\begin{aligned}
\dot{W}_k(t) + \dot{W}_{k+1}(t) &= -\frac{1}{2}\mathbf{G}^T(s,t)\mathbf{P}^2(s)\mathbf{G}(s,t)\,\Big|_{s=L_{sk}}^{L_{fk}^-} - \frac{1}{2}\mathbf{G}^T(s,t)\mathbf{P}^2(s)\mathbf{G}(s,t)\,\Big|_{s=L_{s(k+1)}}^{L_{f(k+1)}^-}\\
&= \frac{1}{2}\mathbf{G}^T(L_{sk},t)\mathbf{P}^2(L_{sk})\mathbf{G}(L_{sk},t) - \frac{1}{2}\mathbf{G}^T(L_{f(k+1)}^-,t)\mathbf{P}^2(L_{f(k+1)}^-,t)\mathbf{G}(L_{f(k+1)}^-,t)\\
&\quad - \frac{1}{2}\tilde{\mathbf{K}}^T(L_{fk}^-,t)\Big[\mathbf{A}^T(L_{fk}^-)\mathbf{P}^2(L_{fk}^-)\mathbf{A}(L_{fk}^-) - \mathbf{A}^T(L_{s(k+1)})\mathbf{P}^2(L_{s(k+1)})\mathbf{A}(L_{s(k+1)})\Big]\tilde{\mathbf{K}}(L_{fk}^-,t)\\
&\leqslant \frac{1}{2}\mathbf{G}^T(L_{sk},t)\mathbf{P}^2(L_{sk})\mathbf{G}(L_{sk},t) - \frac{1}{2}\mathbf{G}^T(L_{f(k+1)}^-,t)\mathbf{P}^2(L_{f(k+1)}^-)\mathbf{G}(L_{f(k+1)}^-,t)\\
&\quad - \|\tilde{\mathbf{K}}(L_{fk},t)\|^2\left[r_c^2(L_{fk}^-)\underline{\sigma}\big(\mathbf{A}(L_{fk}^-)\big)^2 - p_{k+1}^2\bar{\sigma}(\Sigma(L_{s(k+1)})\mathbf{V}d(L_{s(k+1)}))^2\right]\\
&\leqslant \frac{1}{2}\mathbf{G}^T(L_{sk},t)\mathbf{P}^2(L_{sk})\mathbf{G}(L_{sk},t) - \frac{1}{2}\mathbf{G}^T(L_{f(k+1)}^-,t)\mathbf{P}^2(L_{f(k+1)}^-)\mathbf{G}(L_{f(k+1)}^-,t)
\end{aligned}$$

by the reset rule for $\mathbf{A}(L_{s(k+1)}) = p_k\mathbf{I}$ in eqn (21).

Proceeding recursively for $k = 1,\cdots,p-1$,

$$\dot{W}(t) \leqslant -\frac{1}{2}\mathbf{G}^T(L,t)\mathbf{P}^2(L)\mathbf{G}(L,t) + \frac{1}{2}\mathbf{G}^T(0,t)(0,t).$$

Under the hypothesis on the boundary conditions that the inlet flow rate on each lane is the same as the desired flow rate and that $\mathbf{V}_c(0,t) = 0$, $\mathbf{G}(0,t) = \mathbf{A}(s)\tilde{\mathbf{K}}(0,t) = 0$, hence,

$$\dot{W}(t) \leqslant -\frac{1}{2}\mathbf{G}^T(L,t)\mathbf{P}^2(L)\mathbf{G}(L,t) \leqslant 0$$

and $W(t) \leqslant W(0)$. Define

$$\alpha := \sqrt{\frac{\sup_s \bar{\sigma}(\mathbf{A}(s)^T\mathbf{P}(s)\mathbf{A}(s))}{\inf_s \underline{\sigma}(\mathbf{A}(s)^T\mathbf{P}(s)\mathbf{A}(s))}}$$

where $\bar{\sigma}(\mathbf{X})$ and $\underline{\sigma}(\mathbf{X})$ are the maximum and minimum singular values $\mathbf{X}$. Then,

$$\| \tilde{K}(\cdot, t) \| \leqslant \alpha \| \tilde{K}(\cdot, 0) \| .$$

Lyapunov stability in the L_2 sense follows.

APPENDIX B

Before we prove Theorem C, let us recall some definitions and identities in differential geometry.

Let μ be a volume form (i.e. a non-zero d-forms on a d dimensional manifold) and χ a vector field on an oriented manifold $\mathcal{H}$, the *divergence* with respect to μ of χ is defined to be (Abraham *et al.*, 1988),

$$div_\mu \chi \mu := \mathcal{L}_\chi \mu. \tag{B1}$$

where $\mathcal{L}_\chi \mu$ is the Lie-derivative of μ with respect to to vector field χ.

Let α be a d-form, and $\chi, \chi_2, \cdots \chi_d$ be d vector fields, the contraction operator $\rfloor$ is defined to be:

$$(\chi\rfloor\alpha)(\chi_2, \cdots, \chi_d) = \alpha(\chi, \chi_2, \cdots, \chi_d).$$

The Cartan's magic formula is a convenient way of expanding the Lie derivative of a differential form. Let α be a d-form, $d > 0$, and χ a vector field,

$$\mathcal{L}_\chi \alpha = \chi\rfloor \mathrm{d}\alpha + \mathrm{d}(\chi\rfloor\alpha). \tag{B2}$$

Let a be a scalar function on $\mathcal{H}$, and χ a vector field on $\mathcal{H}$,

$$div_\mu a\chi = a div_\mu \chi + L_\chi a, \tag{B3}$$

where $\mathcal{L}_\chi$ denotes the Lie-derivative operation by χ.

Let μ_1, μ_2 be two volume forms. Then μ_1 and μ_2 are related by a non-zero function. Let $div_\mu \chi$ denote the divergence of χ with respect to the volume form μ, and a a non-zero real valued function, then,

$$div_{a\mu} \chi = div_\mu \chi + \frac{1}{a}\mathcal{L}_\chi a. \tag{B4}$$

Proof of Theorem C. The dynamics of the density error is given by eqn (36):

$$\frac{\partial}{\partial t}\tilde{K} = -div_\mu(\tilde{K}V_d) - div_\mu(KV_c), \tag{B5}$$

where $\tilde{K}(x, t) = K(x, t) - K_d(x, t)$ is the density error and $K(x, t) \geqslant 0$ is the actual density.

Define $\mu_1 := \frac{1}{p}\mu$ as a new volume form on $\mathcal{H}$ where $p : \mathcal{H} \to \mathcal{R}^+$ is a positive function as defined in eqn (39).

Consider the Lyapunov functional,

$$W(t) := \frac{1}{2}\int_{\mathcal{H}} \tilde{K}^2(x, t)p(x)\mu.$$

The time derivative is:

$$\begin{aligned} \dot{W} &= -\int (\tilde{K} div_\mu \tilde{K}V_d + \tilde{K} div_\mu(KV_c))p\mu \\ &= -W_1 - W_2, \end{aligned}$$

where

$$W_1 = \int_{\mathcal{H}} p\tilde{K} div_\mu(\tilde{K}V_d)\mu, \qquad W_2 = \int_{\mathcal{H}} p\tilde{K} div_\mu(KV_c)\mu.$$

Using the divergence formulae (B3) and (B4),

$$W_1 = \int p^2\tilde{K}(div_{\mu_1}(\tilde{K}V_d)\frac{1}{p}\mathcal{L}_{\tilde{K}V_d}p)\mu_1$$
$$= \int p^2\tilde{K}\left(\tilde{K}div_{\mu_1}V_d + \mathcal{L}_{V_d}\tilde{K} + \frac{1}{p}\tilde{K}\mathcal{L}_{V_d}p\right)\mu_1.$$

Since $div_{\mu 1}V_d = 0$ in the construction of $p(x)$ in eqn (38),

$$W_1 = \int (p^2\tilde{K}\mathcal{L}_{V_d}\tilde{K} + p\tilde{K}^2\mathcal{L}_{V_d}p)\mu_1$$
$$= \int (p\tilde{K}(\mathcal{L}_{V_d}(p\tilde{K}) - \tilde{K}\mathcal{L}_{V_d}p) + p\tilde{\mathbf{K}}^2\mathcal{L}_{V_d}p)\mu_1$$
$$= \int (p\tilde{K}\mathcal{L}_{V_d}(p\tilde{K}))\mu_1$$
$$= \frac{1}{2}\mathcal{L}_{V_d}(p\tilde{K})^2\mu_1.$$

Consider next W_2 (i.e. the contribution due to $V_c(x,t)$).

$$W_2 = \int_{\mathcal{H}} p\tilde{K}div_\mu(KV_c)_\mu$$
$$= \int \{div_\mu(pK\tilde{K}V_c) - \mathcal{L}_{V_c}(p\tilde{K})\}\mu$$
$$= \int \{div_\mu(pK\tilde{K}V_c) - K(\mathbf{d}(p\tilde{K})\cdot V_c)\}\mu.$$

Using the definition of div in eqn (B1) and the Cartan's magic formula (B2):

$$div_\mu\chi_\mu := \mathcal{L}_\chi\mu = \mathrm{d}[\chi\rfloor_\mu] + \chi\rfloor\mathbf{d}\mu$$
$$= \mathbf{d}\chi\rfloor_\mu$$

since d of any form of top degree is 0.

Thus,

$$\begin{aligned} W_2 &= \int_{\mathcal{H}} \mathbf{d}(pK\tilde{K}V_c)\rfloor\mu - \int_{\mathcal{H}} K(\mathbf{d}h\cdot V_c)\mu \\ &= \int_{\partial\mathcal{H}} pK\tilde{K}(V_c\rfloor\mu) - \int_{\mathcal{H}} K(\mathbf{d}h\cdot V_c)\mu. \end{aligned} \tag{B6}$$

We claim that if V_c is parallel to the boundary on $\partial\mathcal{H}$,

$$\int_{\partial\mathcal{H}} (pK\tilde{K}V_c)\rfloor\mu = 0. \tag{B7}$$

To see this, suppose r is a vector parallel to the boundary, and $V_c = ar$ for some scalar a, then eqn (B7) is an integral of terms in the form:

$$r\rfloor(pK\tilde{K}V_c)\rfloor\mu = pK\tilde{K}\mu(V_c, r) = apK\tilde{K}\mu(r,r) = 0.$$

The second term in eqn (B6) is non-negative since V_c is defined such that $\mathbf{d}(p\tilde{K})\cdot V_c \leqslant 0$.

Thus,

$$\dot{W} = -W_1 - W_2$$
$$\leqslant -\frac{1}{2}\int_{\mathcal{H}} \mathcal{L}_{V_d}(p\tilde{K})^2\mu_1.$$

This expression can be integrated along the flow lines of V_d. Since the flow lines of V_d are assumed to be either closed, or enter and exit at the boundary,

$$\dot{W} \leqslant \frac{1}{2}\int_{\text{inlet}} (p\tilde{K})^2_\mu - \frac{1}{2}\int_{\text{outlet}} (p\tilde{K})^2\mu_1.$$

Moreover, under the hypothesis on the inlet that the flow field is the same as the desired flow field and V_c is parallel to the $\partial\mathcal{H}$,

$$\dot{W} \leqslant -\frac{1}{2}\int_{\text{outlet}} (p\tilde{K})^2\mu_1 = 0.$$

Hence,

$$0 \leqslant W(t) = \frac{1}{2}\int \tilde{K}^2(x,t)p(x)\mu \leqslant W(0),$$

and $\| K(\cdot, t) \|_2 \leqslant \alpha \| K(\cdot, 0) |_2$ with $\alpha = \sup\ p/\inf p$. L_2 stability follows.