Control of Smart Exercise Machines—Part I: Problem Formulation and Nonadaptive Control

Perry Y. Li, *Member, IEEE*, and Roberto Horowitz, *Member, IEEE*

***Abstract*— This is the first part of a two-part paper on the design of intelligent controllers for a class of exercise machines. The control objective is to cause the user to exercise in a manner which optimizes a criterion related to the user's mechanical power. The optimal exercise strategy is determined by a biomechanical behavior of the individual user, which is assumed to satisfy an affine force–velocity relationship dependent on the body geometric configuration. Consequently, the control scheme must simultaneously: 1) identify the user's biomechanical behavior; 2) optimize the controller; and 3) stabilize the system to the estimated optimal states. Moreover, to ensure that the exercise machine is safe to operate, the control system guarantees that the interaction between the exercise machine and the user is passive. In this first part of the paper, we formulate the control problem and propose a controller structure which satisfies the safety requirement and is capable of causing the user to execute an arbitrary exercise strategy *if* the user's biomechanical behavior is known. The controller is of the form of a dynamic damper and can be implemented using only passive mechanical components. Part II of this paper is concerned with the self-optimization problem, in which both the determination of the optimal exercise strategy and the execution of that strategy, when the user's biomechanical behavior is unknown, must be considered.**

***Index Terms*—Adaptive control, biomechanics, hybrid systems, intelligent control, passivity, robotics, self-optimization, velocity field control.**

I. Introduction

In this paper, and the companion paper [1], the design of intelligent control systems for a class of exercise machines to enable the user to exercise in an optimal fashion is considered. The overall system has the following features.

1) It identifies the biomechanical behavior of the user in the form of a relationship between the force, velocity, and position of the exercising limbs for a given level of exertion.
2) It determines, based on the user's force–position–velocity relationship, an optimal exercise strategy for the individual user. A typical optimality criterion used is the mechanical power exerted by the user on the machine at any position. By optimizing this criterion, the user would be able to accomplish a level of exercise in a short amount of time.
3) The control system is able to mechanically resist or assist the user in such a way that the user actually executes the optimal exercise.
4) The interaction between the exercise machine and the user is safe, regardless of what the user does. For our purpose, the exercise control system is "*safe*" if the mechanical interaction with the user is passive. In other words, the net flow of energy must be in the direction from the user to the exercise machine at all times.
5) It adapts to and optimizes the workout for many users with different physical conditioning and motivation during each exercise session.

In this paper, we formulate the intelligent exercise machine control problem and propose a basic controller structure for this problem. To ensure that the machine is safe to operate, the controller structure must be such that the mechanical interaction between the machine and the user is passive. Since the optimal exercise strategy depends on the biomechanical behavior of the user, which varies from person to person and between exercise sessions, the controller must be capable of causing the user to execute an arbitrary exercise strategy (including the optimal strategy). In this paper, the relevant biomechanical behavior is modeled as a nonlinear force–velocity–position relationship, which we refer to as the Hill surface. Each exercise strategy will be encoded as a velocity field on the configuration space of the exercise machine system. The velocity field which encodes the optimal exercise strategy can be specified once the Hill surface has been identified. An interesting feature of this problem is that, in order to track some desired velocity fields, the controller must be capable of assisting the user and of injecting energy in certain portions of the exercise motion cycle, while maintaining passivity of the closed-loop system. The solution proposed in this paper combines a passive velocity field controller [2], [3] with a nonlinear damping term. It ensures that the interaction between the exercise machine under closed-loop control and the user is passive and, under the assumption that the user's Hill surface is known *a priori*, enables the user to execute an arbitrarily specified exercise strategy. One interesting aspect of this controller structure is that it can be implemented as a network of purely passive mechanical components, such as a set of tunable dampers and a mechanical spring. Thus, the exercise machine control system can potentially be manufactured cost effectively as independent add-on modules for many exercise machines already available on the market.

The assumption that the biomechanical behavior of the user is known *a priori* will be relaxed in the companion

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P. Y. Li is with the Department of Mechanical Engineering, University of Minnesota, Minneapolis, MN 55455 USA (e-mail: pli@me.umn.edu).

R. Horowitz is with the Department of Mechanical Engineering, University of California, Berkeley, CA 94720-1740 USA (e-mail: horowitz@me.berkeley.edu).

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paper [1], therefore, the system can optimize the workout of many different users. In [1], the basic control scheme presented in the present paper will be made adaptive and a self-optimizing control scheme will be developed based on a finite-state machine supervisor. The major challenge in self-optimizing control problems (i.e., control systems that optimize an objective criterion for unknown plants), such as the present problem, is that, typically, one cannot determine if the system is operating optimally unless the plant is well known. To determine the optimal behavior of the closed-loop system, the unknown system has to be identified, and this necessitates that the system perform suboptimally during the identification process. This classic conflict, which is sometimes referred to as *dual control* [4], will be resolved in our context in Part II of this paper [1].

Exercise machines are a billion-dollar industry in the U.S., with a wide variety of product offerings. These machines can be classified according to the following characteristics [5].

1) Source of the Exercise Resistance: An individual exercises by moving against some resistive force. The resistance of most exercise machines can be categorized as weight based or as damper based. The resistance is generated by the gravity on a weight stack in weight-based machines and by the damping force in damper-based machines. Exercises performed on damper-based machines are usually concentric (i.e., the user always generates positive power), so that the risk of uncontrolled eccentric stretching can be alleviated.

2) Exercise Motions: The exercise motion determines which muscles are to be exercised. For example, the exercise cycle exercises the muscles in the legs, and the weightlifting machine exercises the arm. Most exercise machines have a single degree of freedom and their exercise motions are repetitive. Exercise motions of machines that have multiple degrees of freedom, e.g., those of a skiing machine, can often be decomposed into multiple independent single-degree-of-freedom motions.

3) Exercise Objectives: Exercise machines cater to different exercise objectives of the users. The two main categories of exercise objectives are: 1) strength building and 2) cardiovascular workout and caloric consumption. Strength-building machines generally prescribe large resistances, whereas those for cardiovascular workout and caloric consumption prescribe smaller resistance, to prevent premature fatigue.

In this paper, we focus on exercise machines with single-degree-of-freedom repetitive exercise motions used for cardiovascular workout and caloric consumption. Our exercise machine control system will aim to maximize the power output of the users throughout the exercise motion, so that the same level of exercise can be achieved in a shorter amount of time. The control system that will be presented has a causality of a generalized mechanical impedance, so it falls into the category of “damper” based resistance machines.

The remainder of this paper is organized as follows. In Section II, we present the model of our prototype experimental exercise machine setup and a simple model of the biomechanics of the user. The objective and the safety requirement of the exercise machine control are also formulated. In Section III, we assume that the parameters for the biomechanical model are known *a priori* and present the design of a novel dynamic damping controller structure, utilizing the PVFC concepts in [2] and [3]. This controller structure satisfies the safety requirement and, at the same time, enables the user to execute an arbitrarily specified exercise strategy. In Section IV, the realization of this controller structure using purely passive mechanical components will be discussed. Section V contains concluding remarks. In Part II of this paper [1], we propose an adaptive version of the controller and a self-optimization strategy to alleviate the requirement that parameters of the biomechanical model of the user are known. Experimental results of the overall control system will also be presented in [1].

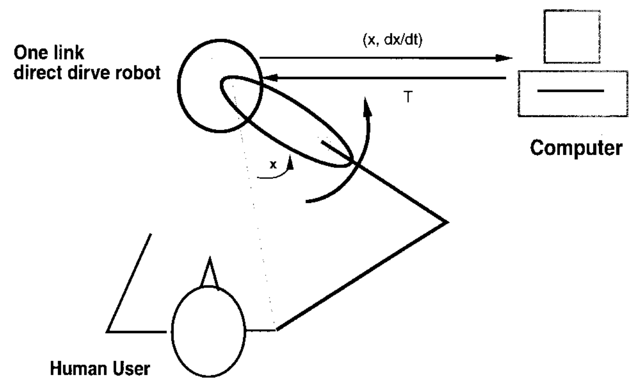

Fig. 1. Experimental exercise machine setup.

II. Modeling and Problem Formulation

A. Dynamic Model of Exercise System

Our goal is to develop a control mechanism which can be used in conjunction with a wide variety of exercise machines, like the ones already on the market. Without loss of generality, we assume that the exercise motion has been designed and consists of a one-degree-of-freedom repetitive movement and that the configuration space $\mathcal{G}$ is the circle S^1. We also assume that the resistive force to movement can be manipulated in real time.

The experimental exercise machine which was used in our research is schematically depicted in Fig. 1. It consists of a rigid link connected to a dc motor. The individual exercising would hold onto the handle attached at the end of the link and rotate the link. The motor in the setup provides for the resistive or assistive force to cause the user to execute the desired exercise strategy. If we assume that the shoulder of the user is fixed while he/she is exercising, the kinematics of the system is described by a four-bar linkage (the rigid link, upper and lower arms, and the imaginary link between the shoulder and the base of the rigid link). The configuration space of this system is a circle and the angle $x \in [0, 2\pi)$ can be used as the coordinate. In this coordinate system, the dynamics of the exercise system take the form

$$M(x(t))\ddot{x}(t) + C(x(t))\dot{x}^2(t) = F(t) + T(t) \qquad (1)$$

where $M(x) \in \Re^+$ is the generalized inertia of the four-bar linkage, $C(x)\dot{x}^2$ represents the Coriolis and centripetal forces,

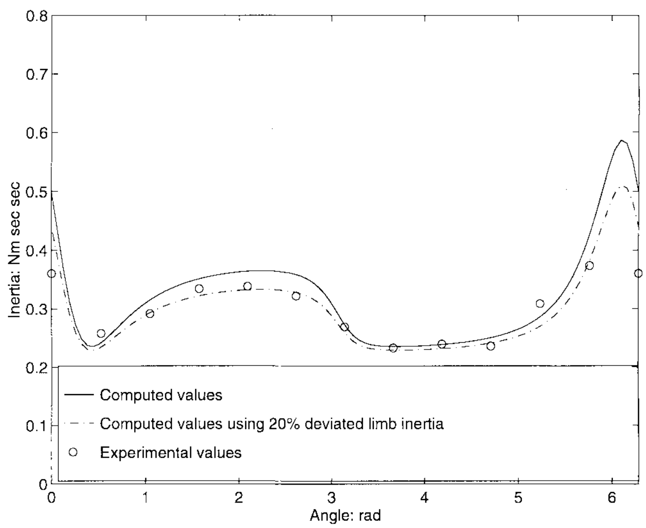

Fig. 2. Estimated inertia functions obtained from short burst experiment and by calculation based on limb inertia data in the literature.

and F and T are the generalized forces generated by the user and the motor, respectively. $M(\cdot)$ and $C(\cdot)$ are related by

$$\frac{d}{dx} M(x) = 2C(x).$$

We shall assume that the function $M(\cdot)$ and, thus, $C(\cdot)$ is known. $M(\cdot)$ can be assembled from the inertia of the exercise machine (the rotor of the motor and the rigid link) and that of the user's upper and lower arms using kinematic relations. The inertia of the user's limbs can be found from regression curves developed from databases (e.g., [6]) where the inertia values of the different body segments are listed as functions of the subject's weight and dimensions. Alternatively, $M(\cdot)$ can be obtained experimentally using a procedure similar to those conducted in [7], where a short burst of force is suddenly applied to the system, and the initial mechanical response is observed before any reflex can take place. We have investigated both methods and have found that the estimates for $M(x)$ are consistent with each other (Fig. 2).

B. Biomechanic Model of the User

We now describe a simple model for the generalized force $F(t)$ in (1) that the user would generate while exercising. First, we briefly review some basic musculoskeletal biomechanics background. The readers are referred to texts such as [8] and [9] for further details.

Muscle fibers are organized into motor units. An electrical impulse activates all the muscle fibers in a motor unit in a binary fashion, resulting in the presence or absence of a twitch. If the impulses arrive at a high-enough frequency, the twitches summate temporally, increasing the generated force. This process of force modulation is termed rate coding. By recruiting and activating many muscle fibers, the total force summate (spatial summation). This method of force modulation is termed recruitment coding. In a normal muscle, both processes are used by the neural system to modulate the total output force.

An activated muscle generates tension along its fibers. This tension varies nonlinearly with the muscle length. A muscle is generally inserted into two or more bones (via tendons) at finite distances from the joint which connects these bones. Thus, when the muscle is in tension, a net internal torque acting on the joint is produced. However, since the muscle length and the moment arm vary with the joint angle, the joint torque produced by the muscle typically varies nonlinearly with the joint angle.

It has been found empirically that, for a given level of electrical stimulation, the force produced by a muscle decreases monotonically with the rate of shortening of the muscle [10]. A hyperbolic relation, known as the Hill relation, has been found to be a good approximation of this relationship over a large range of velocities for individual muscle fibers and for whole muscles. This phenomenon has been attributed to the asymmetric rate of cross-bridge attachment and detachment as the muscle shortens.

We are now in a position to summarize the factors that affect the force that a muscle exerts on the exercise machine.

- *Effort level*—Muscle activation is under voluntary control and, to a lesser extent, depends on the reflexes. This is done by a combination of rate coding and recruitment coding, as explained previously.
- *Position dependence*—As the configuration of the exercise machine varies, the leverages that a joint torque has on the machine changes. Moreover, at different configurations, the joint angles also change. This in turn affects the joint torque by affecting the lever arm that the muscle has around the joint, as well as by changing the effective length of the muscle. The net effect is that the generalized force exerted by the user on the exercise machine depends nonlinearly on the generalized position of the exercise motion.
- *Velocity dependence*—As the joint torque decreases with the shortening velocity of muscle, the force exerted on the exercise machine also decreases with the velocity of motion according to the force–velocity relation, or the Hill relation. A typical force–velocity relationship is shown in Fig. 3. This shows that a muscle will be more capable of generating a large force against a stationary constraint than when the muscle is shortening.
- *Fatigue*—Fatigue is defined to be the decrease in the capability of the muscle to produce force. Thus, for the same level of activation, the force exerted on the exercise machine also decreases with fatigue. The mechanism of fatigue is still an active area of research in muscle physiology.

Most exercise motions involve more than one joint, and each joint is actuated by more than one muscle. The total generalized force of the user [i.e., $F(t)$ in (1)] actuating the exercise motion is, therefore, the sum of the generalized forces exerted by each muscle. Since each of the factors listed above is applicable to the individual muscles, they also affect the total generalized force in the same qualitative manner. In particular, the total generalized force would decrease as the velocity of motion increases and would vary nonlinearly with

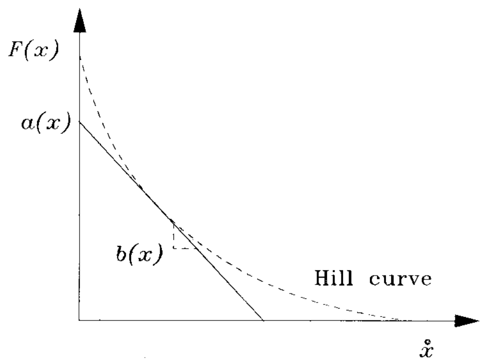

Fig. 3. Force–velocity (Hill) relation of an activated muscle.

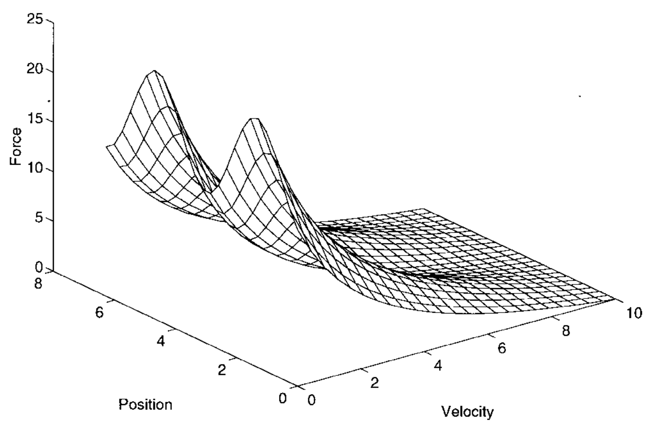

Fig. 4. A typical Hill surface.

the joint angles. For this reason, all the muscles that actuate the exercise motion are grouped and collectively referred to as the *equivalent muscle*.

If the user maintains the same effort level and fatigue state, then the generalized force $F(t)$ in (1) exerted by the equivalent muscle to actuate the exercise motion is described by

$$F(t) = F_h(x(t), \dot{x}(t)) \tag{2}$$

where the function $F_h(x, \dot{x}, t)$ is nonlinear with respect to the position x and decreases monotonically with the velocity of the exercise motion $\dot{x}$. If the effort level and fatigue state vary with time, the function $F_h(\cdot, \cdot)$ will also be time varying. This generalized force–position–velocity relation of the equivalent muscle $F_h(x(t), \dot{x}(t))$ describes a surface

$$\{(x, \dot{x}, F): F = F_h(x, \dot{x})\} \subset \Re^3 \tag{3}$$

which will be referred to as the Hill surface (Fig. 4).

In our application, we assume that the user maintains the same effort level and fatigue state. Furthermore, we make the simplification that the equivalent muscle force decreases *linearly* with the velocity of motion. Thus, for an effort level and a fatigue state, the total equivalent muscle force is modeled by

$$F_h(x, \dot{x}) = a(x) - b(x)\dot{x} \tag{4}$$

where the argument x in the positive functions $a(\cdot)$ and $b(\cdot)$ expresses the dependence of the force–velocity relations on the geometry of the muscle with respect to the configuration of the exercise motion. Notice that $a(\cdot)$ and $b(\cdot)$ vary from user to user and depend on the effort level and the fatigue state of the equivalent muscle.

C. Optimal Exercise

Depending on the dynamics given in (1), the exercise performed by the user is given by a trajectory

$$t \mapsto (x(t), \dot{x}(t), F_h(x(t), \dot{x}(t))$$

which lies on the Hill surface (3) in the space of positions, velocities, and forces. Each trajectory defines the specific manner in which the user exercises. A family of criteria will now be defined to evaluate and compare these trajectories. Since the level of muscle activation (effort level) is voluntary and the mechanism of fatigue is not yet well known, these factors are assumed to be held constant when comparing the efficacies of the exercise trajectories. The family of criteria investigated in this paper are the *modified power criteria*, in which the function

$$J_\rho(x, F, \dot{x}) = F\dot{x}^\rho \tag{5}$$

is to be maximized at all times, subject to the constraint that the equivalent muscle's generalized force F and the generalized velocity $\dot{x}$ satisfy the Hill surface (4). ρ in (5) is a predefined positive constant. Because of the Hill surface, (4) is monotonic with respect to velocity, $\max J_\rho(x, \cdot, \cdot)$ exists for each positive ρ and for each position x. The motivations for these criteria are as follows. By choosing $\rho = 1$, $J_\rho(F, \dot{x})$ is the mechanical power generated by the equivalent muscle. Hence, when $J_{\rho=1}$ is maximized, intuitively, the user must consume energy at a high rate compared to other exercises, enhancing cardiovascular workout and caloric consumption. If $\rho > 1$, the optimal exercise biases toward a lower force/higher velocity condition, whereas, if $\rho < 1$, the exercise is biased toward a higher force/lower velocity condition. It is known in the literature [5] that low-force/high-velocity exercises favor endurance training and high-force/low-velocity exercises favor strength training. Therefore, ρ allows the user to emphasize either endurance or strength training and, at the same time, to consume energy at a high rate.

Assuming that (4) is satisfied, the optimal condition at each x is given by

$$V^*(x) = \frac{\rho}{\rho+1}\frac{a(x)}{b(x)}; \qquad F^*(x) = \frac{a(x)}{\rho+1}. \tag{6}$$

Notice that the optimal exercising condition, $V^*(x)$ in (6) is a *velocity field* on the configuration space of the exercise system, i.e., $V^*(x)$ defines an optimal velocity at each position x.

D. Safety Requirement

Since our exercise machine setup in Fig. 1 includes a motor, it is important that the exercise machine, while under closed-loop control, is safe to operate, regardless of the behavior of the user. For our purpose, we specify safety by requiring that the controlled exercise machine appear like a passive device to the user.

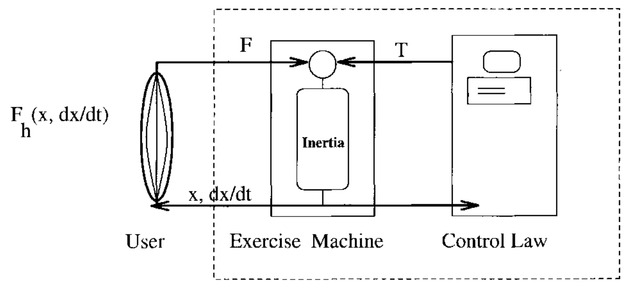

Fig. 5. Exercise machine under closed-loop control.

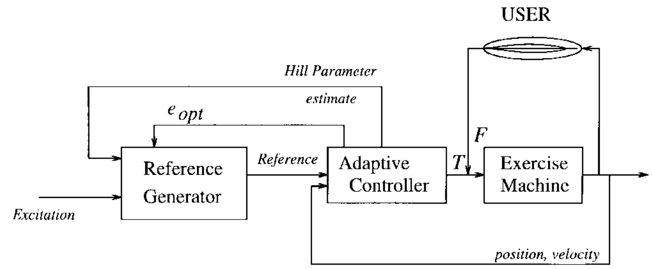

Fig. 6. Control scheme for a smart exercise machine.

Viewing the user's force $F(t)$, and the velocity $\dot{x}(t)$ as the input and output of the controlled exercise machine system, define the supply rate [11] to be $F(t)\dot{x}(t)$, which is the power input into the system. The exercise machine is said to be *passive with respect to the supply rate* $F(t)\dot{x}(t)$ if

$$\int_0^t F(\tau)\dot{x}(\tau)\,d\tau \geq -c^2 \tag{7}$$

for all $t \geq 0$ and *any* human force $F(\tau)$ not necessarily satisfying (4) and some $c \in \Re$.

When (7) is satisfied, the net energy exchange between the user and the machine necessarily flows from the user to the machine, except for the finite initial energy c^2.

E. Control Objectives

The control objectives of the exercise machine control system are summarized as follows.

1) The first is to cause the velocity $\dot{x}$ to follow the velocity field $V^*(x)$, i.e.,

$$\dot{x}(t) \to V^*(x(t)).$$

Notice that $V^*(x)$ is unknown *a priori*, since it depends on the individual's biomechanics, which are unknown.

2) The controlled exercise machine (Fig. 5) is passive with respect to the supply rate $F(t)\dot{x}(t)$, i.e., for any user's force $F(\cdot)$ and any $t \geq 0$, the passivity relationship in (7) must be satisfied.

The stated control problem is a self-optimizing control problem, since the desired velocity field that the control system must follow is a not defined *a priori*, but is rather a function of the unknown user biomechanics. The control system must first identify this optimal desired velocity field and then cause the user to execute it. The following design methodology will be adopted for the design of such a system (Fig. 6):

1) develop a controller capable of causing the user to exercise according to an arbitrary (possibly time-varying) velocity field, assuming that the user's muscular biomechanical behavior $F_h(\cdot, \cdot)$ is known, while ensuring that the interaction between the closed-loop exercise machine and the user is passive to satisfy the safety requirement;
2) formulate a certainty equivalence adaptive control scheme based on the controller developed in step 1), to identify the user's muscular biomechanical behavior and to control the exercise system;
3) construct a reference generator and a finite-state machine supervisor, so that the optimal velocity field is accurately determined and followed by the adaptive controller developed in step 2).

The solution to the first step will be the topic of this paper, while the other two steps will be discussed in the companion paper [1].

III. Exercise Machine Nonadaptive Control

In this section, we develop a controller which ensures that the closed-loop control system is passive with respect to the supply rate $F(t)\dot{x}(t)$ and enables the user to execute an arbitrarily specified possibly time-varying velocity field $V_d(x, t)$ when the Hill surface $F_h(x, \dot{x})$ in (2) is known. The following assumption is made about the Hill surface $F_h(x, \dot{x})$ and the desired velocity field $V_d(x, t)$.

Assumption 1:

1) At each position $x \in \mathcal{G}$, the Hill surface is nonincreasing with respect to the velocity of motion, i.e.,

$$F_h(x, V_1) \leq F_h(x, V_2), \qquad \text{if } V_1 \geq V_2.$$

2) The desired velocity field $V_d(\cdot, \cdot)$ is smooth, positive, and uniformly upper bounded, i.e., for each $x \in \mathcal{G}$, there exists $\overline{V}_d(x)$ such that

$$\overline{V}_d(x) \geq V_d(x, t) > 0, \qquad \text{for all } t \geq 0.$$

3) Let $F_d(x, t) := F_h[x, V_d(x, t)]$ be the force that the user would exert while exercising according to $V_d(x, t)$. There exists $P_{\min} > 0$, so that

$$F_d(x, t)V_d(x, t) \geq P_{\min} > 0 \qquad \forall t \geq 0 \text{ and } x \in \mathcal{G}.$$

Thus, the desired exercise specified by $V_d(x, t)$ is concentric and requires the user to exert strictly positive power.

An intuitive way to design controllers for mechanical systems that guarantee the closed-loop passivity is to mimic the behavior of a static damper,

$$T(t) = -B(x(t), t)\dot{x}, \qquad B(x, t) \geq 0$$

so that the controller would always dissipate energy at the rate $B(x(t), t)\dot{x}^2$. Unfortunately, it is sometimes necessary for the controller to inject energy into the system to assist the user in tracking some desired velocity fields. In this case, $B(x, t)$ cannot be positive everywhere, and the closed loop system can no longer be guaranteed to be passive. Indeed, controllers of the form of a time-invariant static damper

$$T(t) = -B(x(t))\dot{x}$$

can only guarantee closed-loop passivity if and only if $B(x) > 0$ for all $x \in \mathcal{G}$ [12, Prop. 6.1.1].

To resolve this difficulty, a *dynamic damping controller* is proposed, which, in addition to having the ability to dissipate energy, can also store and release energy. It does not, however, generate any on its own. This controller is constructed in two stages. In the first stage, dynamics are introduced into the controller so as to mimic an energy storage element like a flywheel or a spring. Then, a coupling control is designed to shuttle energy conservatively between the energy storage element and the exercise machine system based on a newly developed passive velocity field control (PVFC) theory [2]. When the user's force is absent, this controller will cause the velocity of the exercise machine to converge to a *multiple* of the desired velocity field, where the multiple is determined by the energy currently present in the system. In the second stage, the interaction with the user is taken into account by designing a damping term to regulate the energy level in the system by dissipating any excess power supplied by the user while exercising.

A. Passive Velocity Field Control (PVFC)

Following the design procedure in [2], we combine the dynamics of the exercise machine control system (1) with the dynamics of an energy storage element like a flywheel,

$$M_2 \dot{v}_2 = T_2 \tag{8}$$

to form an augmented mechanical system:

$$\mathbf{M}_a(x)\dot{\mathbf{v}}_a + \mathbf{C}_a(x, \mathbf{v}_a)\mathbf{v}_a = \mathbf{T}_a + \mathbf{F}_a \tag{9}$$

where

$$\mathbf{M}_a(x) := \begin{bmatrix} M(x) & 0 \\ 0 & M_2 \end{bmatrix}; \quad \mathbf{C}_a(x, \mathbf{v}_a) := \begin{bmatrix} C(x)\dot{x} & 0 \\ 0 & 0 \end{bmatrix}$$
$$\mathbf{T}_a := \begin{pmatrix} T \\ T_2 \end{pmatrix}; \quad \mathbf{F}_a := \begin{pmatrix} F \\ 0 \end{pmatrix}; \text{ and } \mathbf{v}_a := \begin{pmatrix} \dot{x} \\ v_2 \end{pmatrix}. \tag{10}$$

In (8), M_2 and v_2 are interpreted to be the inertia and the velocity of a flywheel. Alternatively, they can, respectively, be considered the inverse of the spring coefficient and the force of a mechanical spring. The flywheel control T_2 will be determined subsequently as part of the control law for the augmented control $\mathbf{T}_a$. The total kinetic energy of the augmented system is defined to be the sum of the kinetic energies of the exercise machine and of the flywheel:

$$\kappa_a(x, \mathbf{v}_a) := \tfrac{1}{2}\mathbf{v}_a^T \mathbf{M}_a(x)\mathbf{v}_a = \tfrac{1}{2}M(x)\dot{x}^2 + \tfrac{1}{2}M_2 v_2^2. \tag{11}$$

The *desired* velocity field $\mathbf{V}_a(x)$ for the augmented system is defined to be

$$\mathbf{V}_a(x, t) = \begin{bmatrix} V_d(x, t) \\ V_{d2}(x, t) \end{bmatrix} \tag{12}$$

where $V_d(x, t)$ is the given desired velocity field which specifies the desired exercise for the user, and $V_{d2}(x, t)$ is the desired velocity field for the flywheel which is to be designed. $\mathbf{V}_a(x, t)$ has to be designed to satisfy the following condition.

Conservation of Energy: There exists a constant $\overline{E} > 0$, so that

$$\overline{E} = \kappa_a(x, \mathbf{V}_a(x, t)) \qquad \text{for all } x \in \mathcal{G} \text{ and } t \geq 0. \tag{13}$$

Once a sufficiently large $\overline{E}$ has been chosen (this is always possible since $V_d(x, t)$ is assumed to be bounded in *Assumption 1* and the configuration space $\mathcal{G}$ is compact), $V_{d2}(x, t)$ is computed from

$$V_{d2}(x, t) = \sqrt{\frac{1}{M_2}\left\{2\overline{E} - M(x)V_d^2(x, t)\right\}}. \tag{14}$$

Notice that the constant $\overline{E}$ is the kinetic energy of the augmented system (9) if both the exercise machine and the flywheel velocities conform to their desired values. Therefore, no external energy input is required to maintain the velocity of the augmented system at the desired level, $\mathbf{v}_a(t) = \mathbf{V}_a(x(t), t)$ for all t.

We now design a coupling control term $\mathbf{T}_{a1}$ which transfers energy between the exercise machine system and the flywheel, so that $\mathbf{v}_a(t) \to \beta\mathbf{V}_a(x(t), t)$ where β is a scalar determined by the amount of energy in the system. To specify this control term, the following quantities are needed.

Momentum:

$$\mathbf{p}_a(x, t) = \begin{bmatrix} M(x)\dot{x} \\ M_2 v_2 \end{bmatrix}. \tag{15}$$

Desired momentum:

$$\mathbf{P}_a(x, t) = \begin{bmatrix} M(x)V_d(x, t) \\ M_2 V_{d2}(x, t) \end{bmatrix}. \tag{16}$$

Covariant derivative of $\mathbf{P}_a(x, t)$:

$$\mathbf{\Delta}_a(x, v_a) = \begin{bmatrix} M(x)\dfrac{\partial V_d}{\partial x}(x, t) + C(x)V_d(x, t) \\ M_2 \dfrac{\partial V_{d2}}{\partial x}(x, t) \end{bmatrix}\dot{x}. \tag{17}$$

Time variation of $\mathbf{P}_a(x, t)$:

$$\mathbf{S}_a(x, t) = \begin{bmatrix} M(x)\dfrac{\partial V_d}{\partial t}(x, t) \\ M_2 \dfrac{\partial V_{d2}}{\partial t}(x, t) \end{bmatrix}. \tag{18}$$

The PVFC coupling control is now defined to be

$$\mathbf{T}_{a1} := \mathbf{G}(x, \mathbf{v}_a, t)\mathbf{v}_a + \gamma\mathbf{R}(x, \mathbf{v}_a, t)\mathbf{v}_a \tag{19}$$

where γ is a positive gain constant and

$$\mathbf{G}(x, \mathbf{v}_a, t) := \frac{1}{2\overline{E}}\left[(\mathbf{\Delta}_a + \mathbf{S}_a)\mathbf{P}_a^T - \mathbf{P}_a(\mathbf{\Delta}_a + \mathbf{S}_a)^T\right] \tag{20}$$
$$\mathbf{R}(x, \mathbf{v}_a, t) := [\mathbf{P}_a\mathbf{p}_a^T - \mathbf{p}_a\mathbf{P}_a^T]. \tag{21}$$

Notice that the matrices $\mathbf{G}(x, \mathbf{v}_a)$ and $\mathbf{R}(x, \mathbf{v}_a)$ in (19) are both 2×2 and skew symmetric.

In these expressions, $\mathbf{V}_a(x, t)$ is the *desired* augmented velocity field in (12); $\mathbf{v}_2 = (\dot{x}, v_2)^T$ is the *actual* augmented velocity vector.

Using the facts that $\mathbf{P}_a(x, t)^T \mathbf{V}_a(x, t) = 2\overline{E}$ and the conservation of energy condition in (13), we obtain after some algebra:

$$\mathbf{M}_a(x)\dot{\mathbf{V}}_a(x(t), t) + \mathbf{C}^a(x, \mathbf{v}_a)\mathbf{V}_a(x, t) = \mathbf{G}(x, \mathbf{v}_a, t)\mathbf{V}_a(x, t) \quad (22)$$

where $\dot{\mathbf{V}}_a(x(t), t)$ denotes $[\dot{x}\partial/\partial x + \partial t]\mathbf{V}_a(x, t)$. Thus, the matrix $\mathbf{G}(x, \mathbf{v}_a, t)$ is responsible for generating the inverse dynamics function necessary to track the desired velocity field $\mathbf{V}_a(x, t)$.

For any scalar $\alpha \in \Re$, let us define the α-velocity field tracking error to be

$$\mathbf{e}_\alpha(t) := \mathbf{v}_a(t) - \alpha \mathbf{V}_a(x(t), t). \quad (23)$$

When $\mathbf{T}_a$ in (9) is determined by a feedback function, the closed-loop exercise machine system is an input/output system where the human force $F(t)$ is the input, and the exercise velocity $\dot{x}$ is the output. The following theorem, taken from [2], states that if $\mathbf{T}_a = \mathbf{T}_{a1}$ in (19), then this input–output system is passive with respect to the human–mechanical power $F(t)\dot{x}(t)$ as the supply rate. Moreover, in the absence of human force, the augmented system velocity $\mathbf{v}_a(t)$ converges to a positive multiple of the desired augmented velocity field $\mathbf{V}_a(x, t)$, i.e., for some scalar $\alpha > 0$, the α-velocity field tracking error in (23), $\mathbf{e}_\alpha(t) \to \mathbf{0}$.

Theorem 1 [2]: Consider the augmented mechanical system given by (9) with $\mathbf{T}_a = \mathbf{T}_{a1}$ in (19).

Passivity: The kinetic energy of the augmented system $\kappa_a(x, \mathbf{v}_a)$ given by (11) satisfies

$$\frac{d}{dt}\kappa_a(x(t), \mathbf{v}_a(t)) = F(t)\dot{x}(t) \quad (24)$$

where $F(t)\dot{x}(t)$ is the mechanical power exerted by the human user. Therefore, the input/output system with the human force $F(t)$ as input and the velocity $\dot{x}$ as the output, is passive with respect to the supply rate $F(t)\dot{x}(t)$.

Tracking: Suppose that the human force $F(t) = 0$ for all $t \geq 0$. The following are true.

1) For any scalar $\alpha > 0$, let $\mathbf{e}_\alpha(t)$ be defined as in (23). Then, $\mathbf{e}_\alpha = 0$ is a Lyapunov stable solution.
2) Define $\beta \in \Re$ to be the positive square root of the ratio between the kinetic energy in the augmented system (11) and the desired energy $\overline{E}$ in (13), so that $\kappa_a(x, \mathbf{v}_a) = \beta^2\overline{E}$. Then, $\mathbf{v}_a(t) \to \beta\mathbf{V}_a(x(t), t)$ exponentially from every initial condition except for the equilibria $\mathbf{v}_a(0) = -\beta V_a(x(0), t)$ which are exponentially unstable. That is, $\mathbf{e}_\beta = 0$, where $\mathbf{e}_\beta$ is the α-velocity field tracking error in (23) with $\alpha = \beta$, is almost globally exponentially stable.

Proof: We give a sketch of the proof here. Interested readers are referred to [2] for more details.

1) Differentiating the kinetic energy function (11) with respect to time, and making use of the fact that the matrix

$$\dot{\mathbf{M}}_a(x(t)) - 2\mathbf{C}_a(x(t), \mathbf{v}_a(t)) - \mathbf{G}(x, \mathbf{v}_a, t) - \gamma\mathbf{R}(x, \mathbf{v}_a, t)$$

is skew symmetric, we obtain (24). Since $\kappa_a(x(t), \mathbf{v}_a(t)) \geq 0$, integrating (24) gives (7) with $c^2 = \kappa_a(x(0), \mathbf{v}_a(0))$.

2) To show that $\mathbf{e}_\alpha = 0$ is Lyapunov stable, consider the Lyapunov function

$$W_\alpha(t) = \tfrac{1}{2}\mathbf{e}_\alpha^T \mathbf{M}_a(x)\mathbf{e}_\alpha. \quad (25)$$

Utilizing (9) and (22),

$$\mathbf{M}_a(x)\dot{\mathbf{e}}_\alpha + [\mathbf{C}_a(x, \mathbf{v}_a) - \mathbf{G}(x, \mathbf{v}_a, t)]\mathbf{e}_\alpha = \gamma\mathbf{R}(x, \mathbf{v}_a, t)\mathbf{v}_a + \mathbf{F}_a(t). \quad (26)$$

Differentiating $W_\alpha(t)$ and making use of (26), the fact that $\dot{\mathbf{M}}_a - 2\mathbf{C}_a - \mathbf{G}$ is skew symmetric, the definition of $\mathbf{R}$ in (19), we obtain, after some manipulations,

$$\begin{aligned}\dot{W}_\alpha(t) &= \gamma\mathbf{e}_\alpha^T\mathbf{R}\mathbf{v}_a = -\gamma\alpha\mathbf{V}_a^T\mathbf{R}\mathbf{v}_a + \mathbf{e}_\alpha F(t)\\ &= -\gamma\alpha[4\kappa_a(x, \mathbf{v}_a)\overline{E} - \mathbf{V}_a^T\mathbf{M}_a\mathbf{v}_a] + \mathbf{e}_\alpha F(t). \quad (27)\end{aligned}$$

Since the quantity in $[\cdot]$ is positive (Schwartz's inequality), the Lyapunov stability of $\mathbf{e}_\alpha = 0$ results when $F(t) = 0$.

3) Let $\beta^2 = \kappa_a(x, \mathbf{v}_a)/\overline{E}$ and suppose that $F(t) = 0$. Therefore, β is constant from (24). Substituting $\alpha = \beta$ in (25),

$$\frac{W_\beta}{\beta} = 2\beta\overline{E} - \mathbf{V}_a^T\mathbf{M}_a\mathbf{v}_a.$$

Define also the function

$$\mu(t) := \frac{1}{2}\left\{1 + \frac{\mathbf{V}_a^T(x(t), t)\mathbf{M}_a(x(t))\mathbf{v}_a(t)}{2\beta\overline{E}}\right\}. \quad (28)$$

Notice that by Schwartz' inequality, $0 \leq \mu(t) \leq 1$. Setting $\alpha = \beta$ in (27) and, after some manipulation, we obtain

$$\dot{W}_\beta = -4\gamma\beta\overline{E}\mu(t) \cdot W_\beta. \quad (29)$$

Moreover, differentiating (28),

$$\frac{d}{dt}\mu(t) = 2\frac{\gamma\beta}{\beta^2}\mu(t)W_\beta(t).$$

Thus, since $\gamma\beta > 0$, $\mu(t)$ is nondecreasing and is strictly positive as long as $\mu(0) \neq 0$. This, together with (29), shows that $W_\beta \to 0$ exponentially as long as $\mu(0) \neq 0$. Finally, observe that the set characterized by $\mu(0) = 0$ consists of those initial conditions such that $\mathbf{v}_a(0) = -\beta\mathbf{V}_a(x(0), 0)$, i.e., $\mathbf{e}_{-\beta} = \mathbf{0}$. Considering $t \to -\infty$ and following the same argument with $\beta \to -|\beta|$, it can indeed be shown that $\mathbf{e}_{-|\beta|} = 0$ is exponentially unstable. ■

Loosely speaking, in the absence of an external human force F, if the augmented system has some initial kinetic energy, then this energy will be conserved and distributed between the exercise machine and the flywheel, so that the velocity of augmented system $\mathbf{v}_a$ will asymptotically follow a multiple (β) of the desired velocity field $\mathbf{V}_a$.

B. Dynamic Damping Controller

Since the control system is passive, it does not generate any energy. Therefore, by *Theorem 1*, in the absence of any energy input by the human user, the velocity of the augmented system only converges to a multiple of the desired velocity field, where the multiple β is determined by the energy level of the system. Since the user is supplying energy to the system continuously while exercising, we now design an additional damping term which regulates the energy level in the system so that $\beta = 1$.

The additional damping term is given by

$$\mathbf{T}_{a2} = -\mathbf{B}(x, t)\mathbf{v}_a \tag{30}$$

where $\mathbf{B}(x, t) \in \Re^{2\times 2}$ is a uniformly positive definite damping matrix, i.e., $\exists \underline{b} > 0$ such that

$$\mathbf{v}_a^T \mathbf{B}(x, t)\mathbf{v}_a \geq \underline{b}[\mathbf{v}_a^T \mathbf{v}_a] \qquad \forall \mathbf{v}_a \in \Re^2$$

with the property that the damping force exactly cancels out the force exerted by the human user when $\mathbf{v}_a(t) = \mathbf{V}_a(x(t), t)$.

Let the human force be modeled by the Hill surface $F_h(x, \dot{x})$ and the desired exercise be specified by the desired velocity field $V_d(x, t)$. The damping matrix $\mathbf{B}(x, t)$ must satisfy

$$\mathbf{B}(x, t)\mathbf{V}_a(x, t) = \begin{bmatrix} F_d(x, t) \\ 0 \end{bmatrix} \tag{31}$$

where $F_d(x, t) := F_h(x, V_d(x, t))$ is the force that the human user would exert while exercising at the desired velocity $V_d(x, t)$. One possibility is to define $\mathbf{B}(x, t)$ as

$$\mathbf{B}(x, t) = \frac{F_d(x,t)}{2\overline{E}} \begin{bmatrix} M(x)V_d(x, t) & M_2 V_{d2}(x, t) \\ -M_2 V_{d2}(x, t) & M_2 V_d(x, t) \end{bmatrix}. \tag{32}$$

Equation (31) is satisfied because of the conservation of energy condition (13). The assumption that

$$F_d(x, t)V_d(x, t) \geq P_{\min} > 0$$

in *Assumption 1* ensures that $\mathbf{B}(x, t)$ is uniformly positive definite. Notice that $F_d(x, t)V_d(x, t) > 0$ (i.e., the desired exercise is concentric) is, in fact, both necessary and sufficient for a positive definite damping matrix $\mathbf{B}(x, t)$ satisfying (31) to exist. Necessity can be shown by observing that if $\mathbf{B}(x, t)$ is positive definite, $\mathbf{V}_a^T \mathbf{B}(x, t)\mathbf{V}_a = F_d(x, t)V_d(x, t) > 0$. Sufficiency is established by choosing $\mathbf{B}(x, t)$ as in (32).

The torque input $\mathbf{T}_a \in \Re^2$ to the augmented system in (9) is now given by

$$\mathbf{T}_a = \mathbf{T}_{a1} + \mathbf{T}_{a2}$$

where $\mathbf{T}_{a1}$ is the PVFC coupling control term in (19) and $\mathbf{T}_{a2}$ is the damping term in (30). The closed-loop dynamics are, therefore, given by

$$\mathbf{M}_a(x)\dot{\mathbf{v}}_a + \mathbf{C}_a(x, \mathbf{v}_a)\mathbf{v}_a = -\mathbf{D}(x, \mathbf{v}_a, t)\mathbf{v}_a + \mathbf{F}_a \tag{33}$$

where

$$\mathbf{D}(x, \mathbf{v}_a, t) = -\mathbf{G}(x, \mathbf{v}_a, t) - \gamma\mathbf{R}(x, \mathbf{v}_a, t) + \mathbf{B}(x, t) \tag{34}$$

with $\mathbf{G}(x, \mathbf{v}_a, t)$ and $\mathbf{R}(x, \mathbf{v}_a, t)$ given in (19) and $\mathbf{B}(x, t)$ being the uniformly positive definite damping matrix which satisfies (31). Notice that $\mathbf{D}(x, \mathbf{v}_a, t)$ is positive definite.

The *dynamic damping controller* is given by

$$\begin{pmatrix} T \\ M_2\dot{v}_2 \end{pmatrix} = -\mathbf{D}(x, \mathbf{v}_a, t)\begin{pmatrix} \dot{x} \\ v_2 \end{pmatrix} \tag{35}$$

where $\mathbf{v}_a = [\dot{x}, v_2]^T$. The inputs to the controller are the measured position x and velocity $\dot{x}$ of the exercise machine, the output is the control torque $T(t)$, and the flywheel velocity v_2 is the internal state.

Define the augmented velocity field tracking error to be

$$\begin{aligned} \mathbf{e}(t) &:= \mathbf{v}_a(t) - \mathbf{V}_a(x(t), t) \\ &:= \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix} = \begin{pmatrix} \dot{x}(t) - V_d(x(t), t) \\ v_2(t) - V_{d2}(x(t), t) \end{pmatrix} \end{aligned} \tag{36}$$

which is the α-velocity field tracking error in (23) with $\alpha = 1$.

The properties of the exercise machine (1) under the control of the dynamic damping controller in (35) are given in the following theorem.

Theorem 2: Let $V_d(x, t)$ be the desired velocity field according to which the user is supposed to exercise, and $F_h(x, \dot{x})$ be the Hill surface which models the human force $F(t)$. Suppose that $V_d(x, t)$ and $F_h(x, t)$ satisfy *Assumption 1*. The closed-loop system given by (33), consisting of the exercise machine (1) and the dynamic damping controller (35), has the following properties.

1) The closed-loop system is passive with respect to the supply rate $F(t)\dot{x}(t)$.
2) Let $\mathbf{e}$ be the velocity field tracking error given in (36). There exists a $\zeta > 0$ such that the Lyapunov function

$$W(t) := \tfrac{1}{2}\,\mathbf{e}(t)^T\mathbf{M}_a(x(t))\mathbf{e}(t) \tag{37}$$

satisfies

$$\dot{W}(t) \leq -\zeta W(t) + e_1(t)\{F(t) - F_d(x(t), t)\} \tag{38}$$

where $e_1(t) = \dot{x}(t) - V_d(x(t), t)$. If $\mathbf{B}(x, t)$ is given by (32), then we can choose

$$\zeta = P_{\min}/\overline{E}$$

where $P_{\min}$, given in *Assumption 1*, is the lower bound of the power that the user would exert while exercising according to $V_d(x, t)$, and $\overline{E}$ is the “desired energy” in (13) for the augmented desired velocity field $\mathbf{V}_a$ in (12).

3) If $F(t) = F_h(x(t), \dot{x}(t))$, then $\mathbf{e} \to 0$ exponentially at a rate of at least 0.5ζ, where ζ is given in item 2).

Proof:

1) To see that the closed-loop system is passive with respect to the supply rate $F(t)\dot{x}(t)$, differentiate the total kinetic energy function (11). Using the fact that

$$\dot{\mathbf{M}}_a(x) - 2\mathbf{C}_a(x, \mathbf{v}_a) - \mathbf{G}(x, \mathbf{v}_a, t) - \gamma\mathbf{R}(x, \mathbf{v}_a, t)$$

is skew symmetric, and $\mathbf{B}(x, t)$ is positive definite, we obtain

$$\frac{d}{dt}\kappa_a(x(t), \mathbf{v}_a(t)) = -\mathbf{v}_a^T\mathbf{B}(x, t)\mathbf{v}_a + F\dot{x} \leq F\dot{x}.$$

Upon integration of this expression, and utilizing the fact that $\kappa_a(x, \mathbf{v}_a) \geq 0$, the passivity relation (7) results.

2) The Lyapunov function in (37) is the same as $W_\alpha(t)$ in (25) with $\alpha = 1$. Thus, following the proof of *Theorem 1*,

$$\dot{W}(t) \leq -\mathbf{e}(t)^T\mathbf{B}(x(t), t)\mathbf{v}_a(t) + \mathbf{e}^T(t)\mathbf{F}_a(t).$$

Since $\mathbf{B}(x, t)\mathbf{V}_a(x, t) = [F_d(x, t), 0]^T$ and $\mathbf{F}_a(t) = [F(t), 0]^T$,

$$\begin{aligned}\dot{W}(t) \leq & -\mathbf{e}^T(t)\mathbf{B}[x(t), t]\mathbf{e}(t) + e_1(t)\{F(t) - F_d(x(t), t)\} \\ \leq & -\zeta W(t) + e_1\{F(t) - F_d(x(t), t)\}\end{aligned} \tag{39}$$

where ζ is any positive constant satisfying the inequality

$$\zeta[\mathbf{v}_a^T\mathbf{M}_a(x)\mathbf{v}_a] \leq [\mathbf{v}_a^T\mathbf{B}(x, t)\mathbf{v}_a] \qquad \forall\, \mathbf{v}_a \in \Re^2, \quad \forall\, x \in \mathcal{G}.$$

Notice that such a $\zeta > 0$ exists because $\mathbf{B}(x, t)$ is uniformly positive definite for all x and t and $\mathbf{M}_a(\cdot)$ is bounded. When $\mathbf{B}(x, t)$ is defined by (32), then

$$\mathbf{e}^T\mathbf{B}(x, t)\mathbf{e} = \frac{F_d(x, t)V_d(x, t)}{\overline{E}}\, W(t) \geq P_{\min}W(t)$$

so we can choose $\zeta = P_{\min}$.

3) If $F(t) = F_h(x(t), \dot{x}(t))$, (39) becomes

$$\dot{W} \leq -\zeta W + e_1\{F_h(x(t), \dot{x}(t)) - F_h(x(t), V_d(x(t), t))\}.$$

Since $F_h(x, \cdot)$ is assumed to be nondecreasing in *Assumption 1* and $e_1(t) = \dot{x}(t) - V_d[x(t), t]$,

$$e_1\{F_h(x(t), \dot{x}(t)) - F_h(x(t), V_d(x(t), t))\} \leq 0.$$

Hence, $\dot{W} \leq -\alpha W$. This shows that $W \to 0$ exponentially at the rate of ζ. Since W is quadratic in $\mathbf{e}$, $||\mathbf{e}|| \to 0$ at a rate of 0.5ζ. ■

An important feature of the dynamic damping controller (35) is that, regardless of whether the Hill surface $F_h(x, \dot{x})$ is an accurate model for the human force $F(t)$, closed-loop passivity is not compromised. Thus, the system would still be safe to operate. Knowledge of the Hill surface, however, is necessary to design the damping matrix $\mathbf{B}(x, t)$ to satisfy (31), so that the user can exercise according to the desired velocity field $V_d(x, t)$. For this reason, a direct adaptive controller will be developed in the companion paper [1]. The relationship in (38) will be instrumental in ensuring the stability of the adaptive controller. In order to specify $V_d(x, t)$ to be the optimal velocity field (6), knowledge of the Hill surface is also necessary. A self-optimizing strategy will be developed in [1] for this purpose.

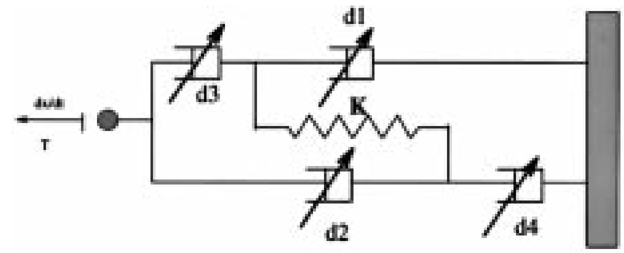

Fig. 7. A realization of the dynamic damping controller using passive mechanical elements.

IV. Controller Realization Using Passive Mechanical Elements

The structure of the dynamic damping controller given by (35) can sometimes be realized using exclusively passive mechanical elements, eliminating the need for an active device like a motor. One advantage of this is that the cost of the motor and the associated amplifier and power supply can be avoided, and the control system can potentially be manufactured in a cost-effective manner as an integrated add-on module suitable for many existing exercise machines. Moreover, since the passivity property becomes an intrinsic property of the hardware, the safety of the system can be preserved, even in the presence of software and computer faults.

We study one realization of the dynamic damping controller (35) where the damping matrix $\mathbf{B}(x, t)$ is designed using (32). This restriction has the property that the off-diagonal entries in $\mathbf{D}(x, \mathbf{v}_a, t)$ in (35) are skew symmetric. The realization makes use of a mechanical spring with spring constant K and four dampers with variable damping coefficients d_1, d_2, d_3, and d_4. It is schematically depicted in Fig. 7, where linear motion components are drawn to highlight the topology. An equivalent mechanism using rotary springs and dampers is necessary to replace the motor in our experimental setup in Fig. 1. A key feature of this realization is that, by differentially choosing the damping coefficients, the spring can be made to either store or to release energy. One possibility for implementing the tunable dampers is to utilize electro-rheological-magnetic (ERM) fluid, the viscosity of which can vary rapidly in response to changes in the applied magnetic field [13].

Let $\mathbf{D}(x(t), \mathbf{v}_a(t), t)$ be the positive definite matrix in (35) with skew–symmetric off-diagonal entries. Denote its ith -row jth-column element by $D_{ij}(t)$.

Assumption 2: There exists a constant $\delta > 0$ such that

$$|D_{12}(t)| \leq \delta \leq \frac{D_{11}(t)D_{22}(t) + D_{12}^2(t)}{|D_{12}(t)|} \tag{40}$$

is satisfied for all t. ■

A sufficient condition for such a δ to exist is

$$\min_t \left[\frac{D_{11}(t)D_{22}(t) + D_{12}^2(t)}{|D_{12}(t)|}\right] > \max_t |D_{12}(t)|. \tag{41}$$

Proposition 1: Let $\mathbf{D}(x, \mathbf{v}_a, t) \in \Re^{2\times 2}$ be the matrix in (35) with skew–symmetric off-diagonal entries. If $\mathbf{D}(x, \mathbf{v}_a, t)$ satisfies *Assumption 2*, then there is a choice for a *constant* spring coefficient $K > 0$ and time-varying *positive* damping coefficients $d_1(t), \cdots, d_4(t)$, such that the dynamics of the mechanism in Fig. 7 duplicate that of the dynamic damping controller in (35).

Proof: Let T be the force output and $\dot{x}$ be the input velocity to the device in Fig. 7 and f be the compressive force of the spring. Let δ be the constant in *Assumption 2* and $f' := \delta f$. The dynamics of the device in Fig. 7 are given by

$$\begin{pmatrix} T \\ \frac{\delta^2}{K} \dot{f}' \end{pmatrix} = -\mathbf{D}^{\dagger}(\delta, d_1, d_2, d_3, d_4)\begin{pmatrix} \dot{x} \\ f' \end{pmatrix}$$

where

$$\mathbf{D}^{\dagger}(\delta, d_1, d_2, d_3, d_4) := \begin{bmatrix} \left(\frac{d_1 d_3}{d_1+d_3} + \frac{d_2 d_4}{d_2+d_4}\right) & \delta\left(\frac{d_2}{d_1+d_3} - \frac{d_3}{d_2+d_4}\right) \\ \delta\left(\frac{d_3}{d_1+d_3} - \frac{d_2}{d_2+d_4}\right) & \delta^2\left(\frac{1}{d_1+d_3} + \frac{1}{d_2+d_4}\right) \end{bmatrix}. \tag{42}$$

Notice that, similar to $\mathbf{D}(x, \mathbf{v}_a, t)$, $\mathbf{D}^{\dagger}()$ is also positive definite for positive $d_1, \cdots, d_4$ and has skew–symmetric off-diagonal entries.

The dynamics of the dynamic damping controller in (35) will be duplicated if $\mathbf{D}^{\dagger}(\delta, d_1, d_2, d_3, d_4) = \mathbf{D}(x, v_a, t)$ and $KM_2 = \delta^2$. To this end, choose the spring constant $K = \delta^2/M_2$ and tune the damping coefficients as follows. Let $D_{ij}(t)$ be the ith-row jth-column element in $\mathbf{D}(x(t), \mathbf{v}_a(t), t)$. If $D_{12}(t) > 0$,

$$d_1(t) = \delta\frac{D_{11}(t)}{|D_{12}(t)|}; \qquad d_2(t) = 0$$
$$d_3(t) = \frac{\delta D_{11}(t)}{\alpha - |D_{12}(t)|}; \qquad d_4(t) = \frac{\delta^2[d_1(t)+d_3(t)]}{D_{22}(t)[d_1(t)+d_3(t)] - \delta^2}.$$

If $D_{12}(t) \le 0$,

$$d_1(t) = \frac{\delta^2[d_2(t)+d_4(t)]}{D_{22}(t)[d_2(t)+d_4(t)] - \delta^2}; \qquad d_2(t) = \frac{\delta D_{11}(t)}{\delta - |D_{12}(t)|}$$
$$d_3(t) = 0; \qquad d_4(t) = \delta\frac{D_{11}(t)}{|D_{12}(t)|}.$$

Substituting these values in (42), we obtain

$$\mathbf{D}^{\dagger}(\delta, d_1(t), d_2(t), d_3(t), d_4(t)) = \mathbf{D}(x(t), v_a(t), t)$$

and $\delta^2/K = M_2$ as required. ∎

The inequality (40) enables positive damping coefficients $d_1, \cdots, d_4$ to be chosen so that $\mathbf{D}^{\dagger}$ in (42) duplicates $\mathbf{D}(.)$ in (35). Although $\delta(t)$ satisfying (41) always exists because $\mathbf{D}(t)$ is positive definite, a constant δ may not. Since δ is used to choose the spring coefficient, *Assumption 2* is important to ensure that a *constant* spring coefficient can be chosen. Notice that (42) is not valid for time-varying spring coefficients.

V. Conclusion

In this paper, we have formulated a control problem for a class of exercise machines that enable the user of the machine to exercise optimally. A Hill surface, relating the force, velocity, and position of the user, was proposed to model the biomechanical behavior of the user. Based on this relationship, an exercise strategy can be selected to optimize a criterion related to the power output of the user. A dynamic damping controller was proposed which consists of a passive velocity field controller and an additional damping term. This controller enforces a passive relationship between the user and the machine, thus ensuring that the machine is safe to operate. When the human biomechanical behavior is known, this controller will also cause the user to execute an arbitrary desired exercise strategy. Thus, the user may exercise optimally if the desired velocity field is selected. The controller can also be realized using exclusively passive mechanical elements. In the companion paper [1], we propose an adaptive version of this controller and a self-optimizing control strategy to identify the unknown user's muscular generalized force–position–velocity relation and to determine and achieve the optimal exercise for that user. Experimental results which verify the efficacy of the overall control system will also be presented.

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Perry Y. Li (S'87–M'89) received the B.A. (Hons.) degree in electrical and information sciences from Cambridge University, Cambridge, U.K., in 1987, the M.S. degree in biomedical engineering from Boston University, Boston, MA, in 1990, and the Ph.D. degree in mechanical engineering from the University of California, Berkeley, in 1995.

In 1995, he joined the Xerox Corporation, Webster, NY, where he performed control system research for xerographic printers and media handling systems and was awarded the Achievement and Special Recognition Awards. Since March 1997, he has been the Nelson Assistant Professor in the Department of Mechanical Engineering, University of Minnesota, Minneapolis. His research interests are in nonlinear and intelligent control systems with applications to man–machine systems, robotics, mechatronics, automated vehicle and highway traffic control, and biomedical devices.

Dr. Li is a member of the American Society of Mechanical Engineers.

Roberto Horowitz (M'89) was born in Caracas, Venezuela, in 1955. He received the B.S. degree with highest honors in 1978 and the Ph.D. degree in mechanical engineering in 1983 from the University of California, Berkeley.

In 1982, he joined the Department of Mechanical Engineering, University of California, Berkeley, where he is currently a Professor. He teaches and conducts research in the areas of adaptive, learning, nonlinear, and optimal control, with applications to micro-electro-mechanical systems (MEMS), computer disk file systems, robotics, mechatronics, and intelligent vehicle and highway systems (IVHS).

Dr. Horowitz was the recipient of an IBM Young Faculty Development Award in 1984 and a National Science Foundation Presidential Young Investigator Award in 1987. He is a member of the American Society of Mechanical Engineers.