OPTIMAL FULL INFORMATION H2 GUARANTEED COST CONTROL OF DISCRETE-TIME SYSTEMS

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ABSTRACT

This paper presents, for discrete-time LTI systems with unstructured dynamic uncertainty, a methodology for designing full information controllers which minimize the upper bound on robust H2 performance given in [1]. It is first shown that this optimal control problem can be cast as a semi-definite program. Then, it is shown that this optimization problem can be solved efficiently and accurately using discrete algebraic Riccati equations.

1 INTRODUCTION

The $\mathcal{H}_2$ norm has long been the most widely-used measure of performance for stable discrete-time LTI systems. There are two reasons why this is the case. From a computational standpoint, the $\mathcal{H}_2$ norm is easy to calculate because it only requires the solution of a single Lyapunov equation and standard matrix manipulations. From an intuitive standpoint, the squared $\mathcal{H}_2$ norm of an LTI system can be interpreted as the trace of the steady-state system output covariance under the assumption that the system is driven by white Gaussian noise with unit covariance. Since many disturbances of interest can be modeled as Gaussian noise (either white or filtered), this makes the $\mathcal{H}_2$ norm a particularly useful measure of performance when the system and its disturbances are well-characterized.

However, it is often the case that the system and/or its disturbances are not well-characterized. In this case, it is customary to model the uncertainty in the system model and express the resulting model as a linear fractional transformation (LFT) of a known state space system and an unknown transfer function with a $\mathcal{H}_\infty$ norm bound which represents the uncertainty in the model.

In this framework, we are interested in determining the worst-case $\mathcal{H}_2$ performance of the discrete-time system over all modeled uncertainty. In general, the unknown part of the system could have some structure, such as in μ-synthesis. Necessary and sufficient conditions for robust $\mathcal{H}_2$ performance in this case are derived in the frequency domain in [2]. The resulting conditions need to be checked at every frequency (or at least a fine grid of frequencies) and then integrated across frequency. In that paper, these conditions are then extended to state space systems and the resulting optimization problem is reduced to a convex optimization problem involving a finite number of linear matrix inequalities (LMIs). However, in both of these approaches, there is a significant amount of conservatism that arises because they do not make any assumptions on the causality of the unknown part of the system.

A related approach for guaranteeing robust performance of a system over model uncertainty is guaranteed cost control [3]. The analysis results of this framework are different than the previous framework in two ways. First, they are time domain analysis results instead of a frequency domain results. Second, the analysis only applies to systems with parametric uncertainty.

In [1], the techniques of guaranteed cost control were used to derive an upper bound on the worst-case $\mathcal{H}_2$ performance of a discrete-time LTI system over unstructured norm-bounded LTI uncertainty. The problem of finding the best $\mathcal{H}_2$ guaranteed cost performance was formulated as a semi-definite program (SDP), which can be solved using solvers such as SeDuMi [4] or by using the `mincx` command in the Robust Control Toolbox for MATLAB. An efficient algorithm for solving this convex optimization problem was then developed which uses the solutions of discrete algebraic Riccati equations (DAREs). This algorithm is analogous to the algorithm developed for continuous-time systems in [5]. It was then shown that the resulting algorithm is faster and tends to be more accurate than using general convex optimization routines to solve the SDP.

This paper considers optimal full information (FI) controller design in terms of this upper bound on robust $\mathcal{H}_2$ performance. As in [1], we will formulate the optimal control design as a SDP and then propose an efficient algorithm for solving this optimization problem using the solutions of discrete algebraic Riccati equations (DAREs). We then show that the resulting algorithm is faster and tends to be more accurate than using general convex optimization routines to solve the SDP.

Throughout the paper, we will use the following notation and definitions. A matrix will be called Schur (resp. anti-Schur) if all of its eigenvalues lie strictly inside (resp. strictly outside) the unit disk in the complex plane. A matrix pair (A,B) will be called d-stabilizable if $\exists K$ such that $A+BK$ is Schur. A matrix pair (A,C) will be called d-detectable if $\exists L$ such that $A+LC$ is Schur. Positive definiteness (resp. semi-definiteness) of a symmetric matrix X will be denoted by $X \succ 0$ (resp. $X \succeq 0$), and a $\bullet$ in a matrix will represent a block which follows from symmetry.

2 PRELIMINARIES

2.1 Inertia of Matrices

We begin by reviewing some basic facts about the inertia of symmetric matrices. For a symmetric matrix, X, we define the functions $\nu_+(X)$, $\nu_0(X)$, and $\nu_-(X)$ to respectively be the number of positive, zero, and negative eigenvalues of X counted with multiplicity. The inertia of the symmetric matrix X is then defined as

$$\mathcal{N}(X) := (\nu_+(X), \nu_0(X), \nu_-(X)).$$

The following result is the fundamental result on the inertia of a symmetric matrix (see, e.g., [6]).

Proposition 2.1. *If X is a symmetric matrix and M is an invertible matrix, then $\mathcal{N}(X) = \mathcal{N}(M^T XM)$.*

Based on this result, we can construct the following two corollaries which will be useful in §4.

Corollary 2.2. *Let X_{11} and X_{22} be symmetric matrices and define $X := \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix}$. Then each of the equations*

$$\mathcal{N}(X) = \mathcal{N}(X_{11} - X_{12}X_{22}^{-1}X_{12}^T) + \mathcal{N}(X_{22})$$
$$\mathcal{N}(X) = \mathcal{N}(X_{22} - X_{12}^T X_{11}^{-1}X_{12}) + \mathcal{N}(X_{11})$$

hold when the relevant inverses exist.

Proof. Define $\Psi_1 := X_{11} - X_{12}X_{22}^{-1}X_{12}^T$, $\Psi_2 := X_{22} - X_{12}^T X_{11}^{-1}X_{12}$,

$$X := \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix}, \quad M_1 := \begin{bmatrix} I & 0 \\ -X_{22}^{-1}X_{12}^T & I \end{bmatrix} \quad M_2 := \begin{bmatrix} I & -X_{11}^{-1}X_{12} \\ 0 & I \end{bmatrix}$$

and note that M_1 and M_2 are invertible. Thus, by Proposition 2.1,

$$\mathcal{N}(X) = \mathcal{N}(M_1^T XM_1) = \mathcal{N}(\mathrm{diag}(\Psi_1, X_{22})) = \mathcal{N}(\Psi_1) + \mathcal{N}(X_{22})$$
$$\mathcal{N}(X) = \mathcal{N}(M_2^T XM_2) = \mathcal{N}(\mathrm{diag}(X_{11}, \Psi_2)) = \mathcal{N}(\Psi_2) + \mathcal{N}(X_{11})$$

which concludes the proof. □

Corollary 2.3. *Let $P = P^T, R = R^T$, assume that $B^T PB + R$ is invertible, and define $\Psi := P - PB(B^T PB + R)^{-1}B^T P$. Then $\mathcal{N}(\Psi) + \mathcal{N}(B^T PB + R) = \mathcal{N}(P) + \mathcal{N}(R)$.*

Proof. Define $X := \begin{bmatrix} P & PB \\ B^T P & B^T PB + R \end{bmatrix}$ and $M := \begin{bmatrix} I & -P^\dagger PB \\ 0 & I \end{bmatrix}$ where $P^\dagger$ denotes the Moore–Penrose pseudoinverse of P. Recall that $PP^\dagger P = P$. Noting that M is invertible, we use Proposition 2.1 to see that $\mathcal{N}(X) = \mathcal{N}(M^T XM) = \mathcal{N}(\mathrm{diag}(P,R)) = \mathcal{N}(P) + \mathcal{N}(R)$. Since, by Corollary 2.2, $\mathcal{N}(X) = \mathcal{N}(\Psi) + \mathcal{N}(B^T PB + R)$ this concludes the proof. □

2.2 Discrete Algebraic Riccati Equations

We first introduce some notation we will be using throughout the paper. For given (A,B,Q,R,S), where $Q = Q^T$ and $R = R^T$, we define

$$\begin{aligned} \mathcal{R}_{(A,B,Q,R,S)}(P) &:= A^T PA + Q \\ &\quad - (A^T PB + S)(B^T PB + R)^{-1}(B^T PA + S^T) \\ \mathcal{A}_{(A,B,Q,R,S)}(P) &:= A - B(B^T PB + R)^{-1}(B^T PA + S^T) \\ \mathcal{L}_{(A,B,Q,R,S)}(P) &:= \begin{bmatrix} A^T PA + Q - P & A^T PB + S \\ \bullet & B^T PB + R \end{bmatrix}. \end{aligned}$$

We will make the notation more compact in the remainder of the paper by respectively denoting these quantities as $\mathcal{R}_\phi(P)$, $\mathcal{A}_\phi(P)$, and $\mathcal{L}_\phi(P)$ where ϕ is an appropriately defined 5-tuple. Note that the equation $\mathcal{R}_\phi(P) = P$ is a DARE. If $\mathcal{R}_\phi(P) = P = P^T$ and $\mathcal{A}_\phi(P)$ is Schur (resp. anti-Schur), then P is called a stabilizing (resp. anti-stabilizing) solution of the DARE.

We now present a few basic results which can be proved using straightforward algebra.

Proposition 2.4. *Let R be invertible and define*

$$\begin{aligned} \phi &:= (A,B,Q,R,S) \\ \bar{\phi} &:= (A - BR^{-1}S^T,\ B,\ Q - SR^{-1}S^T,\ R,\ 0). \end{aligned}$$

Then $\mathcal{R}_\phi(P) = \mathcal{R}_{\bar{\phi}}(P)$ and $\mathcal{A}_\phi(P) = \mathcal{A}_{\bar{\phi}}(P)$. Moreover, $\mathcal{A}_\phi(P) = (I + BR^{-1}B^T P)^{-1}(A - BR^{-1}S^T)$.

Proposition 2.5. *Let R be invertible and define $\phi := (A,B,Q,R,S)$. If $\mathcal{R}_\phi(P) = P$ then $\mathcal{A} = \mathcal{A}_\phi(P)$ satisfies*

$$\begin{bmatrix} A - BR^{-1}S^T & 0 \\ -Q + SR^{-1}S^T & I \end{bmatrix} \begin{bmatrix} I \\ P \end{bmatrix} = \begin{bmatrix} I & BR^{-1}B^T \\ 0 & (A - BR^{-1}S^T)^T \end{bmatrix} \begin{bmatrix} I \\ P \end{bmatrix} \mathcal{A}. \qquad (1)$$

Conversely, if $A - BR^{-1}S^T$ is invertible and $P, \mathcal{A}$ satisfy Eq. (1)*, then $\mathcal{R}_\phi(P) = P$ and $\mathcal{A}_\phi(P) = \mathcal{A}$.*

With these two propositions in place, we can now state and prove the main result of this subsection, which relates a class of DARE solutions to a class of LMIs.

Theorem 2.6. *Let Q and R be invertible and define*

$$\begin{aligned} \phi &:= (A,\ B,\ C^TQ^{-1}C,\ R,\ 0) \\ \psi &:= (A^T,\ C^T,\ -BR^{-1}B^T,\ -Q,\ 0). \end{aligned}$$

If the DARE $\mathcal{R}_\phi(P) = P$ has a stabilizing solution P_0, then $P_0 \prec P$ for any $P \succ 0$ which satisfies $\mathcal{L}_\psi(P^{-1}) \prec 0$.

Proof. We first make the additional assumptions that A and P_0 are invertible; we will relax these assumptions in the final part of the proof. For convenience, we define $X_0 := P_0^{-1}$. Noting that $\mathcal{A}_\phi(P_0) = (I + BR^{-1}B^TP_0)^{-1}A$ (by Proposition 2.4), we see that $\mathcal{A}_\phi(P_0)$ is invertible. By Proposition 2.5,

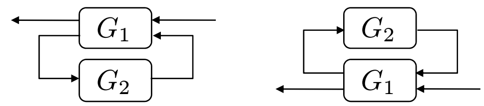
$$\begin{aligned} &\begin{bmatrix} A & 0 \\ -C^TQ^{-1}C & I \end{bmatrix} \begin{bmatrix} I \\ P_0 \end{bmatrix} = \begin{bmatrix} I & BR^{-1}B^T \\ 0 & A^T \end{bmatrix} \begin{bmatrix} I \\ P_0 \end{bmatrix} \mathcal{A}_{\hat\phi}(X_0) \\ \Rightarrow \quad &\begin{bmatrix} A^T & 0 \\ BR^{-1}B^T & I \end{bmatrix} \begin{bmatrix} I \\ X_0 \end{bmatrix} = \begin{bmatrix} I & -C^TQ^{-1}C \\ 0 & A \end{bmatrix} \begin{bmatrix} I \\ X_0 \end{bmatrix} \bar{\mathcal{A}} \end{aligned}$$

where $\bar{\mathcal{A}} = X_0^{-1}(\mathcal{A}_\phi(X_0))^{-1}X_0$. Since $\bar{\mathcal{A}}$ and $(\mathcal{A}_\phi(X_0))^{-1}$ are related by a similarity transformation, we see that $\bar{\mathcal{A}}$ is anti-Schur. Thus, since A^T is nonsingular, we apply Proposition 2.5 again to see that X_0 is the anti-stabilizing solution of the DARE $\mathcal{R}_\psi(X) = X$.

Let $P \succ 0$ satisfy $\mathcal{L}_\psi(P^{-1}) \prec 0$. We will now show that $P_0 \prec P$. For convenience we define $X := P^{-1} \succ 0$. By Schur complements, $CXC^T - Q \prec 0$ and $\mathcal{R}_\psi(X) - X \prec 0$. With some algebra, it can be shown that

$$\begin{aligned} &(X_0 - X) - \bar{\mathcal{A}}^T(X_0 - X)\bar{\mathcal{A}} \\ &\qquad = (\mathcal{R}_\psi(X) - X) + (L_0 - L)\left(CXC^T - Q\right)(L_0 - L)^T \end{aligned}$$

where $L_0 := -AX_0C^T(CX_0C^T - Q)^{-1}$ and $L := -AXC^T(CXC^T - Q)^{-1}$. Since the right-hand side of this equation is negative definite and $\bar{\mathcal{A}}$ is anti-Schur, we conclude by Lyapunov equation theory that $X_0 - X \succ 0$. Thus, $0 \prec X \prec X_0 \Rightarrow 0 \prec P_0 \prec P$.



(a) LOWER LFT (b) UPPER LFT

Figure 1: LINEAR FRACTIONAL TRANSFORMATIONS

We now relax the invertibility assumptions that we made at the beginning of the proof. We will show that $P_0 \prec P$ by perturbing P_0 by a small amount to produce P_ε which satisfies $P_0 \prec P_\varepsilon \prec P$. Note that, for $\Delta \succ 0$ with sufficiently large minimum eigenvalue, $\exists \tilde{C}$ such that $\Delta - P_0\mathcal{A}_\phi(P_0) - (\mathcal{A}_\phi(P_0))^TP_0 = \tilde{C}^T\tilde{C} \succ 0$. Choose such values of Δ and $\tilde{C}$. We now define

$$A_\varepsilon := A + \varepsilon I, \qquad \hat{C}_\varepsilon := \begin{bmatrix} C \\ \sqrt{\varepsilon}\tilde{C} \end{bmatrix}, \qquad \hat{Q} := \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix},$$

$$\begin{aligned} \phi_\varepsilon &:= (A_\varepsilon,\ B,\ \hat{C}_\varepsilon^T\hat{Q}^{-1}\hat{C}_\varepsilon,\ R,\ 0), \\ \psi_\varepsilon &:= (A_\varepsilon^T,\ \hat{C}_\varepsilon^T,\ -BR^{-1}B^T,\ -\hat{Q},\ 0). \end{aligned}$$

Note that $\mathcal{L}_{\psi_\varepsilon}(X)|_{\varepsilon=0} = \begin{bmatrix} \mathcal{L}_\psi(X) & 0 \\ 0 & -I \end{bmatrix} \prec 0$. Thus, for sufficiently small $\varepsilon > 0$, $\mathcal{L}_{\psi_\varepsilon}(X) \prec 0$. Also note that $\phi_\varepsilon|_{\varepsilon=0} = \phi$. Since the stabilizing solution of a DARE is analytic in its parameters [7], $\exists \bar\varepsilon > 0$ such that $\forall \varepsilon \in (-\bar\varepsilon, \bar\varepsilon)$, the DARE $\mathcal{R}_{\phi_\varepsilon}(X) = X$ has a stabilizing solution P_ε and, moreover, P_ε is an analytic function of ε. Note in particular that $P_\varepsilon|_{\varepsilon=0} = P_0$. Implicitly differentiating the DARE $\mathcal{R}_{\phi_\varepsilon}(P_\varepsilon) = P_\varepsilon$ with respect to ε and denoting the derivative of P_ε as P'_ε, we obtain after some algebra that

$$\begin{aligned} P'_\varepsilon &= \left(\mathcal{A}_{\phi_\varepsilon}(P_\varepsilon)\right)^T P'_\varepsilon \left(\mathcal{A}_{\phi_\varepsilon}(P_\varepsilon)\right) \\ &\quad + P_\varepsilon(\mathcal{A}_{\phi_\varepsilon}(P_\varepsilon)) + (\mathcal{A}_{\phi_\varepsilon}(P_\varepsilon))^TP_\varepsilon + \tilde{C}^T\tilde{C} \\ \Rightarrow \quad P'_\varepsilon\big|_{\varepsilon=0} &= (\mathcal{A}_\phi(P_0))^T \left(P'_\varepsilon\big|_{\varepsilon=0}\right)(\mathcal{A}_\phi(P_0)) + \Delta. \end{aligned}$$

Since $\mathcal{A}_\phi(P_0)$ is Schur and $\Delta \succ 0$, we see by Lyapunov equation theory that $P'_\varepsilon|_{\varepsilon=0} \succ 0$. Thus, $\exists \hat\varepsilon > 0$ such that A_ε and P_ε are invertible and $\mathcal{L}_{\psi_\varepsilon}(X) \prec 0$, $\forall \varepsilon \in (0, \hat\varepsilon)$. By the first part of the proof, $0 \prec P_\varepsilon \prec P$, $\forall \varepsilon \in (0, \hat\varepsilon)$. Therefore, since $P'_\varepsilon|_{\varepsilon=0} \succ 0$, we obtain that $P_0 \prec P_\varepsilon \prec P$ for sufficiently small $\varepsilon > 0$. □

2.3 Analysis of Robust H2 Performance

Let G_1 and G_2 be causal, finite dimensional LTI systems. In this paper, we will make frequent use of the linear fractional transformations (LFTs) shown in Fig. 1. We will notate the lower LFT of G_1 by G_2 as $\mathcal{F}_l(G_1, G_2)$ and the upper LFT of G_1 by G_2 as $\mathcal{F}_u(G_1, G_2)$. The $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms of G_1 will be denoted respectively by $\|G_1\|_\infty$ and $\|G_1\|_2$. We now turn our attention to

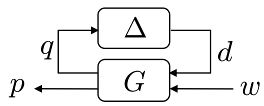


Figure 2: SYSTEM STRUCTURE FOR ANALYSIS

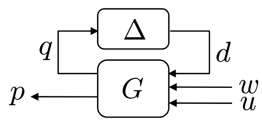


Figure 3: SYSTEM STRUCTURE FOR CONTROL

analyzing the interconnection Fig. 2 where G has the realization

$$G \sim \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \tag{2}$$

and $\|\Delta\|_\infty \leq 1$. We now define the functions

$$\begin{aligned}
&\mathcal{M}(W,V,P,\tau,G) := \\
&\begin{bmatrix} P & 0 & V \\ \bullet & \tau I & 0 \\ \bullet & \bullet & W \end{bmatrix} - \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix}^T \begin{bmatrix} P & 0 & 0 \\ \bullet & \tau I & 0 \\ \bullet & \bullet & I \end{bmatrix} \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} \\
&J_\tau(G) := \inf_{W,V,P} \operatorname{tr}\{W\} \quad \text{s.t.} \quad P \succ 0,\ \mathcal{M}(W,V,P,\tau,G) \succ 0 \\
&J(G) := \inf_{\tau>0} J_\tau(G).
\end{aligned}$$

It can be shown for invertible T that $P \succ 0$, $\mathcal{M}(W,V,P,\tau,G) \succ 0 \Leftrightarrow T^T PT \succ 0$, $\mathcal{M}(W,VT,T^T PT,\tau,\hat{G}) \succ 0$, where $\hat{G}$ and G are equivalent realizations related by the state transformation $x_{\hat{G}} = T^{-1}x_G$. This means that the values of $J_\tau(G)$ and $J(G)$ are not affected by changing the realization of G.

The following proposition summarizes a few useful properties of $J(G)$ and $J_\tau(G)$.

Proposition 2.7. *If G by given by Eq.* (2)*, then*

1. $J(G) \geq \sup_\Delta \|\mathcal{F}_u(G,\Delta)\|_2^2$ *subject to* $\|\Delta\|_\infty \leq 1$.
2. $J(G) \neq \infty \Leftrightarrow$ *the interconnection in Fig. 2 is robustly stable.*
3. *If* $J_\tau(G) \neq \infty$*, then* $J_{\alpha\tau}(G) \neq \infty$, $\forall \alpha \geq 1$.
4. *Define* $B_{1,\tau} := \frac{1}{\sqrt{\tau}} B_1$,

$$\begin{aligned}
\left[\, C_\tau \middle| D_{1,\tau} \middle| D_{2,\tau} \,\right] &:= \left[\begin{array}{c|c|c} \sqrt{\tau} C_1 & D_{11} & \sqrt{\tau} D_{12} \\ C_2 & \frac{1}{\sqrt{\tau}} D_{21} & D_{22} \end{array}\right], \\
\phi &:= (A,\ B_{1,\tau},\ C_\tau^T C_\tau,\ D_{1,\tau}^T D_{1,\tau} - I,\ C_\tau^T D_{1,\tau}), \\
\omega &:= (B_2,\ B_{1,\tau},\ D_{2,\tau}^T D_{2,\tau},\ D_{1,\tau}^T D_{1,\tau} - I,\ D_{2,\tau}^T D_{1,\tau}).
\end{aligned}$$

If the DARE $\mathcal{R}_\phi(P) = P$ *has a stabilizing solution* P_0 *such that* $B_{1,\tau}^T P_0 B_{1,\tau} + D_{1,\tau}^T D_{1,\tau} - I \prec 0$*, then* $J_\tau(G) = \mathcal{R}_\omega(P_0)$*. If not, then* $J_\tau(G) = \infty$*.*

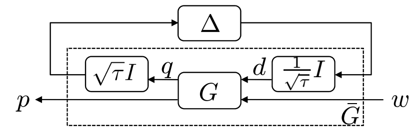


Figure 4: UNCERTAINTY SCALING

5. $J_\tau(G) \neq \infty \Leftrightarrow \left\| \left[\begin{array}{c|c} A & B_1^\tau \\ \hline C^\tau & D_1^\tau \end{array}\right] \right\|_\infty < 1.$
6. $J_\tau(G) = J_1(\bar{G})$ *where* $\bar{G}$ *is as depicted in Fig. 4.*

Proof. Statements 1–3 are explicitly proven in [1]. Statement 4 is a trivial restatement of the result in [1]. Statement 5 is proved by noting that the solvability of the DARE in statement 4 is equivalent to the relevant $\mathcal{H}_\infty$ norm condition. Statement 6 is proved by noting that the expressions in statement 4 are the same for $J_\tau(G)$ and $J_1(\bar{G})$. □

3 SDP APPROACH TO FI CONTROL

In this section, we consider the optimal control (in terms of J) of the interconnection shown in Fig. 3, where it is assumed that the control, u, is generated by a controller which has causal access to d, w, and the state of G. For this paper, we will restrict the controller to lie in the set $\mathcal{K}$, which we define to be the set of controllers which are LTI and have finite order. We also define the set $\mathcal{K}_0$, which only contains controllers which are a static gain. To simplify notation, we will introduce the realization

$$G_{FI} \sim \left[\begin{array}{c|ccc} A & B_1 & B_2 & B_3 \\ \hline C_1 & D_{11} & D_{12} & D_{13} \\ C_2 & D_{21} & D_{22} & D_{23} \\ \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{array}\right] \tag{3}$$

so that the closed loop performance can be written $J(\mathcal{F}_l(G_{FI},K)))$, where $K \in \mathcal{K}$. (G in Fig. 3 corresponds to only considering the first two outputs of G_{FI}.) The dimensions of the signals w, d, and u are respectively n_w, n_d, and n_u.

We begin by proving that optimizing over dynamic controllers is equivalent to optimizing over static controllers.

Theorem 3.1. *If* G_{FI} *is given by Eq.* (3) *and we define*

$$\gamma := \inf_{K \in \mathcal{K}} J(\mathcal{F}_l(G_{FI},K)), \qquad \gamma_0 := \inf_{\bar{K} \in \mathcal{K}_0} J(\mathcal{F}_l(G_{FI},\bar{K}))$$

then $\gamma = \gamma_0$.

Proof. Since $\mathcal{K}_0 \subset \mathcal{K}$, we trivially have the inequality $\gamma \leq \gamma_0$. Thus, it only remains to prove that $\gamma \geq \gamma_0$. Since this is trivial if

γ is infinite, we assume that γ is finite. Fix $\varepsilon > 0$ and then choose $K \in \mathcal{K}$ so that $J(\mathcal{F}_l(G_{FI},K)) < \gamma + \varepsilon$. Now let K and $\mathcal{F}_l(G_{FI},K)$ respectively have the realizations

$$K \sim \left[\begin{array}{c|ccc} A^K & B_1^K & B_2^K & B_3^K \\ \hline C^K & D_1^K & D_2^K & D_3^K \end{array}\right]$$

$$\Rightarrow \mathcal{F}_l(G_{FI},K) \sim \left[\begin{array}{cc|cc} A & 0 & B_1 & B_2 \\ 0 & 0 & 0 & 0 \\ \hline C_1 & 0 & D_{11} & D_{12} \\ C_2 & 0 & D_{21} & D_{22} \end{array}\right] + \left[\begin{array}{cc} 0 & B_3 \\ I & 0 \\ \hline 0 & D_{13} \\ 0 & D_{23} \end{array}\right] \left[\begin{array}{cc|cc} B_1^K & A^K & B_2^K & B_3^K \\ D_1^K & C^K & D_2^K & D_3^K \end{array}\right]$$
$$=: \left[\begin{array}{c|cc} A^{cl} & B_1^{cl} & B_2^{cl} \\ \hline C_1^{cl} & D_{11}^{cl} & D_{12}^{cl} \\ C_2^{cl} & D_{21}^{cl} & D_{22}^{cl} \end{array}\right]$$

and choose $\tau > 0, P \succ 0, W, V$ such that $\mathrm{tr}(W) < \gamma + \varepsilon$ and $\mathcal{M}(W,V,P,\tau,\mathcal{F}_l(G_{FI},K)) \succ 0$. We now partition P and V respectively as $\begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}$ and $\begin{bmatrix} V_1 & V_2 \end{bmatrix}$ where P_{11} and V_1 have as many columns as A and define $\bar{K} := [D_1^K - C^K P_{22}^{-1} P_{12}^T \;\; D_2^K \;\; D_3^K] \in \mathcal{K}_0$, $\bar{P} := P_{11} - P_{12} P_{22}^{-1} P_{12}^T$, and $\bar{V} := V_1 - V_2 P_{22}^{-1} P_{12}^T$. Note that, by Schur complements (applied to P), $\bar{P} \succ 0$. It can also be shown that $\mathcal{M}(W,\bar{V},\bar{P},\tau,\mathcal{F}_l(G_{FI},\bar{K})) \succ 0$. Thus, $J_\tau(\mathcal{F}_l(G_{FI},\bar{K})) \leq \gamma + \varepsilon$, which in turn implies that $\gamma_0 \leq J(\mathcal{F}_l(G_{FI},\bar{K})) \leq \gamma + \varepsilon$. Since the choice of ε was arbitrary, we conclude that $\gamma \geq \gamma_0$. □

With this theorem in place, we can now formulate the problem of finding the best controller as an optimization problem.

Theorem 3.2. *Solving* $\inf_{(\varepsilon>0, K\in\mathcal{K})} J_{(\varepsilon^{-1})}(\mathcal{F}_l(G_{FI},K))$ *is equivalent to solving*

$$\inf_{W,\hat{V},Q,\varepsilon,\hat{K}_x,\hat{K}_d,K_w} \mathrm{tr}\{W\} \quad s.t. \tag{4a}$$

$$\begin{bmatrix} Q & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & \varepsilon I & \bullet & \bullet & \bullet & \bullet \\ \hat{V} & 0 & W & \bullet & \bullet & \bullet \\ AQ + B_3\hat{K}_x & \varepsilon B_1 + B_3\hat{K}_d & B_2 + B_3 K_w & Q & \bullet & \bullet \\ C_1 Q + D_{13}\hat{K}_x & \varepsilon D_{11} + D_{13}\hat{K}_d & D_{12} + D_{13} K_w & 0 & \varepsilon I & \bullet \\ C_2 Q + D_{23}\hat{K}_x & \varepsilon D_{21} + D_{23}\hat{K}_d & D_{22} + D_{23} K_w & 0 & 0 & I \end{bmatrix} \succ 0. \tag{4b}$$

Moreover, for any feasible iterate of the latter optimization problem, the controller

$$K = \begin{bmatrix} \hat{K}_x Q^{-1} & \varepsilon^{-1}\hat{K}_d & K_w \end{bmatrix} \tag{5}$$

achieves the performance $J(\mathcal{F}_l(G_{FI},K)) < \mathrm{tr}\{W\}$.

Proof. By Theorem 3.1, we make the restriction (without any loss in closed loop performance) $K \in \mathcal{K}_0$. We now let K have the form $K = [K_x \;\; K_d \;\; K_w]$ and note that

$$\mathcal{F}_l(G_{FI},K) \sim \left[\begin{array}{c|cc} A + B_3 K_x & B_1 + B_3 K_d & B_2 + B_3 K_w \\ \hline C_1 + D_{13} K_x & D_{11} + D_{13} K_d & D_{12} + D_{13} K_w \\ C_2 + D_{23} K_x & D_{21} + D_{23} K_d & D_{22} + D_{23} K_w \end{array}\right].$$

Note that $J_{(\varepsilon^{-1})}(\mathcal{F}_l(G_{FI},K)) < \gamma \Leftrightarrow \exists P \succ 0, W, V$ such that $\mathrm{tr}\{W\} \leq \gamma$, $\mathcal{M}(W,V,P,\varepsilon^{-1},\mathcal{F}_l(G_{FI},K)) \succ 0$. By Schur complements, this is equivalent to $\exists W,V,P$ such that $\mathrm{tr}\{W\} \leq \gamma$ and

$$\begin{bmatrix} P & \bullet & \bullet & \bullet & \bullet & \bullet \\ 0 & \varepsilon^{-1} I & \bullet & \bullet & \bullet & \bullet \\ V & 0 & W & \bullet & \bullet & \bullet \\ A + B_3 K_x & B_1 + B_3 K_d & B_2 + B_3 K_w & P^{-1} & \bullet & \bullet \\ C_1 + D_{13} K_x & D_{11} + D_{13} K_d & D_{12} + D_{13} K_w & 0 & \varepsilon I & \bullet \\ C_2 + D_{23} K_x & D_{21} + D_{23} K_d & D_{22} + D_{23} K_w & 0 & 0 & I \end{bmatrix} \succ 0.$$

Since the matrix $\Phi := \mathrm{diag}(P^{-1}, \varepsilon I, I, I, I, I) = \Phi^T$ is invertible, we multiply the preceding matrix inequality on the left and right by Φ to see that these conditions are in turn equivalent to $\exists W,V,P$ such that $\mathrm{tr}\{W\} \leq \gamma$ and Eq. (4b) holds, where $\hat{V} := VP^{-1}$, $Q := P^{-1}$, $\hat{K}_x := K_x P^{-1}$, and $\hat{K}_d := \varepsilon K_d$. (Note that K can be reconstructed from $\hat{K}_x, \hat{K}_d$, and K_w using Eq. (5).) Thus, $J_{(\varepsilon^{-1})}(\mathcal{F}_l(G_{FI},K)) = \inf_{W,V,P} \mathrm{tr}\{W\}$ subject to Eq. (4b). Since optimizing over K is equivalent to optimizing over $\hat{K}_x, \hat{K}_d$, and K_w, we see that the the two optimizations are equivalent. □

If the strict inequality is relaxed to a non-strict inequality in the preceding theorem, Eq. (4) becomes a SDP. Thus, a reasonable way to solve the optimal FI control problem is a relax Eq. (4) to a SDP, solve the SDP using an appropriate solver, then reconstruct the controller using Eq. (5).

4 DARE APPROACH TO FI CONTROL

In the previous section, we showed how to perform the optimization $\inf_{K\in\mathcal{K}} J(\mathcal{F}_l(G_{FI},K))$ using a SDP. In this section, we instead solve the equivalent problem

$$\inf_{\varepsilon>0} \inf_{K\in\mathcal{K}} J_{(\varepsilon^{-1})}(\mathcal{F}_l(G_{FI},K)). \tag{6}$$

It should be noted that, since simultaneously performing both optimizations is a convex optimization problem, performing the optimizations sequentially are both convex optimization problems [8]. In particular, this is useful because, given a method for quickly solving the inner optimization problem, the remaining optimization over $\varepsilon > 0$ is convex.

In this section, we will first show that the inner optimization problem can be solved using a single DARE when $\varepsilon = 1$. Then, using this result, we show that the inner optimization problem can be solved using a single DARE for any fixed value of $\varepsilon >$

0. Finally, we present a methodology for solving Eq. (6) which exploits this structure.

A few quantities which will be important in this section are

$$\begin{aligned}
B_{[1,3]} &:= \begin{bmatrix} B_1 \ B_3 \end{bmatrix},\\
\begin{bmatrix} C \middle| D_3 \end{bmatrix} &:= \begin{bmatrix} C_1 & D_{13}\\ C_2 & D_{23} \end{bmatrix},\\
\begin{bmatrix} Q_\varepsilon \middle| S_\varepsilon \end{bmatrix} &:= C_1^T \begin{bmatrix} C_1 \middle| D_{11} \ D_{13} \end{bmatrix} + \varepsilon C_2^T \begin{bmatrix} C_2 \middle| D_{21} \ D_{23} \end{bmatrix},\\
\begin{bmatrix} \bar{Q}_\varepsilon \middle| \bar{S}_\varepsilon \end{bmatrix} &:= D_{12}^T \begin{bmatrix} D_{12} \middle| D_{11} \ D_{13} \end{bmatrix} + \varepsilon D_{22}^T \begin{bmatrix} D_{22} \middle| D_{21} \ D_{23} \end{bmatrix},\\
R_\varepsilon &:= \begin{bmatrix} D_{11}^T D_{11} - I & D_{11}^T D_{13}\\ \bullet & D_{13}^T D_{13} \end{bmatrix} + \varepsilon \begin{bmatrix} D_{21}^T D_{21} & D_{21}^T D_{23}\\ \bullet & D_{23}^T D_{23} \end{bmatrix},\\
\phi_\varepsilon &:= (A,\ B_{[1,3]},\ Q_\varepsilon,\ R_\varepsilon,\ S_\varepsilon),\\
\omega_\varepsilon &:= (B_2,\ B_{[1,3]},\ \bar{Q}_\varepsilon,\ R_\varepsilon,\ \bar{S}_\varepsilon).
\end{aligned} \tag{7}$$

The assumptions and notation that we will be using at various points in the section are:

(A1) G_{FI} is given by Eq. (3) and the notation in Eq. (7) is used
(A2) $D_3^T D_3$ is invertible
(A3) (A, B_3) is d-stabilizable
(A4) $(\hat{A}, \hat{C})$ is d-detectable, where

$$\begin{aligned}
\hat{A} &:= A - B_3(D_{13}^T D_{13} + \varepsilon D_{23}^T D_{23})^{-1}(D_{13}^T C_1 + \varepsilon D_{23}^T C_2)\\
\hat{C} &:= C - D_3(D_{13}^T D_{13} + \varepsilon D_{23}^T D_{23})^{-1}(D_{13}^T C_1 + \varepsilon D_{23}^T C_2).
\end{aligned}$$

4.1 Optimizing J_1

In this subsection, we consider the special case of optimizing J_1, i.e. optimizing $J_{(\varepsilon^{-1})}$ when $\varepsilon = 1$. We will begin by stating (without proof) a standard proposition involving matrix inverses.

Proposition 4.1. *Let M_{33} be an invertible matrix and define* $\begin{bmatrix} \bar{M}_{11} & \bar{M}_{12}\\ \bar{M}_{21} & \bar{M}_{22} \end{bmatrix} := \begin{bmatrix} M_{11} & M_{12}\\ M_{21} & M_{22} \end{bmatrix} - \begin{bmatrix} M_{13}\\ M_{23} \end{bmatrix} M_{33}^{-1} \begin{bmatrix} M_{31} & M_{32} \end{bmatrix}$. *Then* $M_{11} - \begin{bmatrix} M_{12} & M_{13} \end{bmatrix} \begin{bmatrix} M_{22} & M_{23}\\ M_{32} & M_{33} \end{bmatrix}^{-1} \begin{bmatrix} M_{21}\\ M_{31} \end{bmatrix} = \bar{M}_{11} - \bar{M}_{12}\bar{M}_{22}^{-1}\bar{M}_{21}$ *whenever either side of the equation is well-defined.*

We will now use the methodology of [9] to simplify Eq. (4) (for $\varepsilon = 1$) to a more convenient form.

Lemma 4.2. *Suppose assumptions (A1)–(A2) hold. Choose U to be a matrix so that its columns form an orthonormal basis for $\ker(D_3^T)$ and define*

$$\begin{gathered}
D_1 := \begin{bmatrix} D_{11}\\ D_{21} \end{bmatrix}, \quad D_2 := \begin{bmatrix} D_{12}\\ D_{22} \end{bmatrix}, \quad D_{[1,3]} := \begin{bmatrix} D_1 \ D_3 \end{bmatrix},\\
\bar{A} := A - B_{[1,3]} R_1^{-1} D_{[1,3]}^T C, \qquad \begin{bmatrix} \bar{C} \ \ \bar{D}_1 \end{bmatrix} := U^T \begin{bmatrix} C \ \ D_1 \end{bmatrix},\\
\psi := (\bar{A}^T,\ \bar{C}^T,\ -B_{[1,3]} R_1^{-1} B_{[1,3]}^T,\ \bar{D}_1 \bar{D}_1^T - I,\ 0).
\end{gathered}$$

If R_1 is invertible, then $\inf_{K\in\mathcal{K}} J_1(\mathcal{F}_l(G_{FI}, K)) = \inf_{W,Q} \operatorname{tr}\{W\}$ subject to $Q \succ 0$, $\mathcal{L}_\psi(Q) \prec 0$, and

$$\begin{bmatrix} W - D_2^T D_2 + D_2^T D_{[1,3]} R_1^{-1} D_{[1,3]}^T D_2 & \bullet\\ B_2 - B_{[1,3]} R_1^{-1} D_{[1,3]}^T D_2 & Q + B_{[1,3]} R_1^{-1} B_{[1,3]}^T \end{bmatrix} \succ 0. \tag{8}$$

If R_1 is not invertible, then $\inf_{K\in\mathcal{K}} J_1(\mathcal{F}_l(G_{FI}, K)) = \infty$.

Proof. It can be shown that $UU^T = I - D_3(D_3^T D_3)^{-1} D_3^T$. We first define for convenience $\bar{D}_2 := U^T D_2$ and

$$\begin{bmatrix} \tilde{A} \ \tilde{B}_1 \ \tilde{B}_2 \end{bmatrix} := \begin{bmatrix} A \ B_1 \ B_2 \end{bmatrix} - B_3(D_3^T D_3)^{-1} D_3^T \begin{bmatrix} C \ D_1 \ D_2 \end{bmatrix}.$$

Similar to [9], the columns of the matrix $\begin{bmatrix} I & 0\\ -D_3(D_3^T D_3)^{-1} B_3^T & U \end{bmatrix}$ form a basis for the null space of the matrix $[B_3^T \ D_3^T]$. Thus, using the methodology of [9] to eliminate the matrix variable $[\hat{K}_x \ \hat{K}_d \ K_w]$ from Eq. (4) (with $\varepsilon = 1$) yields the equivalent optimization problem $\inf_{W,\hat{V},Q} \operatorname{tr}\{W\}$ subject to

$$\begin{bmatrix} Q & \bullet & \bullet & \bullet & \bullet\\ 0 & I & \bullet & \bullet & \bullet\\ \hat{V} & 0 & W & \bullet & \bullet\\ \tilde{A}Q & \tilde{B}_1 & \tilde{B}_2 & Q + B_3(D_3^T D_3)^{-1} B_3^T & \bullet\\ \bar{C}Q & \bar{D}_1 & \bar{D}_2 & 0 & I \end{bmatrix} \succ 0, \qquad \begin{bmatrix} Q & 0\\ 0 & I \end{bmatrix} \succ 0.$$

Note that since the second of these inequalities is redundant, we are left with just the first matrix inequality. Using this methodology a second time, we then eliminate $\hat{V}$ to yield the equivalent optimization problem $\inf_{W,Q} \operatorname{tr}\{W\}$ subject to

$$\begin{gathered}
\begin{bmatrix} I & \bullet & \bullet & \bullet\\ 0 & W & \bullet & \bullet\\ \tilde{B}_1 & \tilde{B}_2 & Q + B_3(D_3^T D_3)^{-1} B_3^T & \bullet\\ \bar{D}_1 & \bar{D}_2 & 0 & I \end{bmatrix} \succ 0\\
\begin{bmatrix} Q & \bullet & \bullet & \bullet\\ 0 & I & \bullet & \bullet\\ \tilde{A}Q & \tilde{B}_1 & Q + B_3(D_3^T D_3)^{-1} B_3^T & \bullet\\ \bar{C}Q & \bar{D}_1 & 0 & I \end{bmatrix} \succ 0.
\end{gathered}$$

By Schur complements, this optimization problem is equivalent to $\inf_{W,Q} \operatorname{tr}\{W\}$ subject to $Q \succ 0$ and

$$\begin{bmatrix} I - \bar{D}_1^T \bar{D}_1 & \bullet & \bullet\\ -\bar{D}_2^T \bar{D}_1 & W - \bar{D}_2^T \bar{D}_2 & \bullet\\ \tilde{B}_1 & \tilde{B}_2 & Q + B_3(D_3^T D_3)^{-1} B_3^T \end{bmatrix} \succ 0 \tag{9a}$$

$$\begin{bmatrix} Q + B_3(D_3^T D_3)^{-1} B_3^T & \bullet\\ 0 & I \end{bmatrix} - \begin{bmatrix} \tilde{A}Q & \tilde{B}_1\\ \bar{C}Q & \bar{D}_1 \end{bmatrix} \begin{bmatrix} Q^{-1} & 0\\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{A}Q & \tilde{B}_1\\ \bar{C}Q & \bar{D}_1 \end{bmatrix}^T \succ 0. \tag{9b}$$

Note that if $I-\bar{D}_1^T\bar{D}_1$ is singular, this optimization problem is infeasible. Since, by Corollary 2.2, R_1 is invertible $\Leftrightarrow$ $\bar{D}_1^T\bar{D}_1-I$ is invertible, we see that the optimization problem is infeasible (i.e. $\inf_{K\in\mathcal{K}} J_1(\mathcal{F}_l(G_{FI},K))=\infty$) if R_1 is not invertible.

For the remainder of the proof, we assume that R_1 is invertible. We now note that, by Proposition 4.1,

$$\begin{aligned}
\bar{A} = A - B_{[1,3]}R_1^{-1}D_{[1,3]}^T C &= \tilde{A} - \tilde{B}_1(\bar{D}_1^T\bar{D}_1-I)^{-1}\bar{D}_1^T\bar{C} \\
B_2 - B_{[1,3]}R_1^{-1}D_{[1,3]}^T D_2 &= \tilde{B}_2 - \tilde{B}_1(\bar{D}_1^T\bar{D}_1-I)^{-1}\bar{D}_1^T\bar{D}_2 \\
B_{[1,3]}R_1^{-1}B_{[1,3]}^T &= \tilde{B}_1(\bar{D}_1^T\bar{D}_1-I)^{-1}\tilde{B}_1^T + B_3(D_3^TD_3)^{-1}B_3^T \\
I - D_{[1,3]}R_1^{-1}D_{[1,3]}^T &= U\left[I+\bar{D}_1(I-\bar{D}_1^T\bar{D}_1)^{-1}\bar{D}_1^T\right]U^T.
\end{aligned} \tag{10}$$

By Eq. (10) and Schur complements, the inequality Eq. (9a) is equivalent to the inequalities $I-\bar{D}_1^T\bar{D}_1\succ 0$ and Eq. (8). We now define the matrix $M_2 := \begin{bmatrix} I & -\tilde{B}_1\bar{D}_1^T(\bar{D}_1\bar{D}_1^T-I)^{-1} \\ 0 & I\end{bmatrix}$ and note that $\bar{D}_1^T(\bar{D}_1\bar{D}_1^T-I)^{-1} = (\bar{D}_1^T\bar{D}_1-I)^{-1}\bar{D}_1^T$. Multiplying Eq. (9b) on the left and right respectively by M_2 and M_2^T and applying Eq. (10) yields with a little algebra that Eq. (9b) is equivalent to $\mathcal{L}_\psi(Q)\prec 0$. Thus Eq. (9) is equivalent to the system of inequalities $\mathcal{L}_\psi(Q)\prec 0$, $I-\bar{D}_1^T\bar{D}_1\succ 0$, and Eq. (8). However, the constraint $I-\bar{D}_1^T\bar{D}_1\succ 0$ is redundant because the constraints $\mathcal{L}_\psi(Q)\prec 0$ and $Q\succ 0$ imply that $I-\bar{D}_1\bar{D}_1^T\succ \bar{C}Q\bar{C}^T\succeq 0$. □

With this lemma in place, we can now state and prove the main result of this subsection, which expresses the optimal controller and its associated cost in terms of a DARE solution.

Theorem 4.3. *Suppose assumptions (A1)–(A2) hold and P_0 is a stabilizing solution of the DARE $\mathcal{R}_{\phi_1}(P)=P$ such that $P_0\succeq 0$ and $\mathcal{N}(B_{[1,3]}^TP_0B_{[1,3]}+R_1)=(n_u,0,n_d)$. Then*

$$\begin{aligned}
&\inf_{\bar{K}\in\mathcal{K}} J_1(\mathcal{F}_l(G_{FI},\bar{K})) = J_1(\mathcal{F}_l(G_{FI},K)) = \operatorname{tr}\{\mathcal{R}_{\omega_1}(P_0)\} \\
&K := \begin{bmatrix} K_x & K_d & K_w\end{bmatrix} \\
&\quad := -(B_3^TP_0B_3+D_3^TD_3)^{-1}\begin{bmatrix} B_3^TP_0 & D_3^T\end{bmatrix}\begin{bmatrix} A & B_1 & B_2 \\ C & D_1 & D_2\end{bmatrix}.
\end{aligned}$$

Proof. We will first show that the controller K achieves the cost $\operatorname{tr}\{\mathcal{R}_{\omega_1}(P_0)\}$. To this end, we define the closed loop state space matrices and tuples

$$\begin{aligned}
\begin{bmatrix} A^{cl} & B_1^{cl} & B_2^{cl} \\ C^{cl} & D_1^{cl} & D_2^{cl}\end{bmatrix} &:= \begin{bmatrix} A & B_1 & B_2 \\ C & D_1 & D_2\end{bmatrix} + \begin{bmatrix} B_3 \\ D_3\end{bmatrix}\begin{bmatrix} K_x & K_d & K_w\end{bmatrix} \\
\phi_{cl} &:= (A^{cl},\, B_1^{cl},\, (C^{cl})^TC^{cl},\, (D_1^{cl})^TD_1^{cl}-I,\, (C^{cl})^TD_1^{cl}) \\
\omega_{cl} &:= (B_2^{cl},\, B_1^{cl},\, (D_2^{cl})^TD_2^{cl},\, (D_1^{cl})^TD_1^{cl}-I,\, (D_2^{cl})^TD_1^{cl}).
\end{aligned}$$

By Proposition 4.1 and a little algebra, $\mathcal{R}_{\phi_1}(P_0)=\mathcal{R}_{\phi_{cl}}(P_0)$, $\mathcal{R}_{\omega_1}(P_0)=\mathcal{R}_{\omega_{cl}}(P_0)$, and $\mathcal{A}_{\phi_1}(P_0)=\mathcal{A}_{\phi_{cl}}(P_0)$. Therefore, P_0 is the stabilizing solution of the DARE $\mathcal{R}_{\phi_{cl}}(P)=P$. Also, since $B_3^TP_0B_3+D_3^TD_3\succeq D_3^TD_3\succ 0$, Corollary 2.2 implies after a little algebra that $(B_1^{cl})^TP_0B_1^{cl}+(D_1^{cl})^TD_1^{cl}-I\prec 0$. Thus, by Proposition 2.7, we see that $J_1(\mathcal{F}_l(G_{FI},K))=\mathcal{R}_{\omega_{cl}}(P_0)=\mathcal{R}_{\omega_1}(P_0)$.

It now only remains to show that the performance achieved by the controller K is optimal. Defining $\Phi := P_0 - P_0B_{[1,3]}(B_{[1,3]}^TP_0B_{[1,3]}+R_1)^{-1}B_{[1,3]}^TP_0$, we see by Corollaries 2.2 and 2.3 that

$$\begin{aligned}
\mathcal{N}(\Phi)+\mathcal{N}(B_{[1,3]}^TP_0B_{[1,3]}+R_1) &= \mathcal{N}(P_0)+\mathcal{N}(R_1) \\
&= \mathcal{N}(P_0)+\mathcal{N}(\bar{D}_1^T\bar{D}_1-I)+\mathcal{N}(D_3^TD_3).
\end{aligned}$$

Since $\nu_+(B_{[1,3]}^TP_0B_{[1,3]}+R_1)=\nu_+(D_3^TD_3)$, we see that $\nu_+(\Phi)\geq\nu_+(P_0)$. Similarly, $\nu_0(\Phi)\geq\nu_0(P_0)$. Thus, since P_0 and Φ have the same dimension, $\mathcal{N}(\Phi)=\mathcal{N}(P_0)$. Therefore, $\mathcal{N}(B_{[1,3]}^TP_0B_{[1,3]}+R)=\mathcal{N}(R)$, which implies that R_1 is invertible, which in turn implies that $\bar{D}_1^T\bar{D}_1-I$ is invertible. We now define $\bar{\phi} := (\bar{A},\, B_{[1,3]},\, \bar{C}^T(I-\bar{D}_1\bar{D}_1^T)^{-1}\bar{C},\, R_1,\, 0)$. Note that, by Proposition 2.4 and the last identity in Eq. (10), P_0 is the stabilizing solution of the DARE $\mathcal{R}_{\bar{\phi}}(P)=P$.

We now choose $Q\succ 0, W$ such that $\mathcal{L}_\psi(Q)\prec 0$ and Eq. (8) holds (where ψ is as defined in Lemma 4.2). By Theorem 2.6, we see that $P_0\prec Q^{-1}$. If P_0 is invertible, we see that $Q\prec P_0^{-1}\Rightarrow 0\prec Q+B_{[1,3]}R_1^{-1}B_{[1,3]}^T\prec P_0^{-1}+B_{[1,3]}R_1^{-1}B_{[1,3]}^T\Rightarrow \Phi=(P_0^{-1}+B_{[1,3]}R_1^{-1}B_{[1,3]}^T)^{-1}\prec(Q+B_{[1,3]}R_1^{-1}B_{[1,3]}^T)^{-1}$. If P_0 is not invertible, replace P_0 by $P_0+\varepsilon I$ for small $\varepsilon>0$, repeat the same arguments, then let $\varepsilon\to 0$ to conclude that $\Phi\preceq(Q+B_{[1,3]}R_1^{-1}B_{[1,3]}^T)^{-1}$. Thus, from Eq. (8),

$$\begin{aligned}
W &\succ D_2^TD_2 - D_2^TD_{[1,3]}R_1^{-1}D_{[1,3]}^T + \bar{B}_2^T(Q+B_{[1,3]}R_1^{-1}B_{[1,3]}^T)^{-1}\bar{B}_2 \\
&\succeq D_2^TD_2 - D_2^TD_{[1,3]}R_1^{-1}D_{[1,3]}^T + \bar{B}_2^T\Phi\bar{B}_2 \\
&= \mathcal{R}_{(\bar{B}_2,\, B_{[1,3]},\, D_2^TD_2-D_2^TD_{[1,3]}R_1^{-1}D_{[1,3]}^T,\, R,\, 0)}(P_0)
\end{aligned}$$

where $\bar{B}_2 := B_2 - B_{[1,3]}R^{-1}D_{[1,3]}^TD_2$. Using Proposition 2.4, we see that the last of these expressions equals $\mathcal{R}_{\omega_1}(P_0)$. Thus, $W\succ\mathcal{R}_{\omega_1}(P_0)\Rightarrow\operatorname{tr}\{W\}>\operatorname{tr}\{\mathcal{R}_{\omega_1}(P_0)\}$. □

In the preceding theorem, we assumed the existence of a DARE solution with several relevant properties. The next theorem gives a set of conditions which guarantee that the DARE solution has a solution with the required properties.

Theorem 4.4. *Suppose (A1)–(A4) hold for $\varepsilon=1$. Then $\exists K\in\mathcal{K}$ such that $J_1(\mathcal{F}_l(G_{FI},K))\neq\infty\Leftrightarrow$ the DARE $\mathcal{R}_{\phi_1}(P)=P$ has a stabilizing solution $P_0\succeq 0$ such that $\mathcal{N}(B_{[1,3]}^TP_0B_{[1,3]}+R_1)=(n_u,0,n_d)$.*

Proof. ($\Leftarrow$) This is trivial by Theorem 4.3.

($\Rightarrow$) By Proposition 2.7, we know that if $J_1(\mathcal{F}_l(G_{FI},K))$ is finite, then $\|\mathcal{F}_l(G_{FI},K)[I_{n_d}\ 0]^T\|_\infty<1$. By standard discrete-time

$\mathcal{H}_\infty$ theory (see, e.g., [10]), this implies that the DARE $\mathcal{R}_{\phi_1}(P) = P$ has a stabilizing solution $P_0 \succeq 0$ such that the factorization

$$B_{[1,3]}^T P_0 B_{[1,3]} + R_1 = \begin{bmatrix} T_{11} & 0 \\ T_{21} & T_{22} \end{bmatrix}^T \begin{bmatrix} -I_{n_d} & 0 \\ 0 & I_{n_u} \end{bmatrix} \begin{bmatrix} T_{11} & 0 \\ T_{21} & T_{22} \end{bmatrix}$$

exists where T_{11} and T_{22} are invertible. By Proposition 2.1, this implies that $\mathcal{N}(B_{[1,3]}^T P_0 B_{[1,3]} + R_1) = (n_u, 0, n_d)$. □

4.2 Optimal FI Control

In this subsection, we use the results of the previous subsection with statement (6) of Proposition 2.7 to solve $\inf_{K \in \mathcal{K}} J_{(\varepsilon^{-1})}(\mathcal{F}_l(G_{FI}, K))$. Based on this, we then propose a methodology to solve Eq. (6).

Theorem 4.5. *Let $\varepsilon > 0$ and suppose that(A1)–(A2) hold. If P_0 is a stabilizing solution of the DARE $\mathcal{R}_{\phi_\varepsilon}(P) = P$ such that $P_0 \succeq 0$ and $\mathcal{N}(B_{[1,3]}^T P_0 B_{[1,3]} + R_\varepsilon) = (n_u, 0, n_d)$, then*

$$\inf_{K \in \mathcal{K}} J_{(\varepsilon^{-1})}(\mathcal{F}_l(G_{FI}, K)) = J_{(\varepsilon^{-1})}(\mathcal{F}_l(G_{FI}, K_\varepsilon)) = \varepsilon^{-1} \operatorname{tr}\{\mathcal{R}_{\omega_\varepsilon}(P_0)\}$$

$$K_\varepsilon := \begin{bmatrix} K_{x,\varepsilon} & K_{d,\varepsilon} & K_{w,\varepsilon} \end{bmatrix} := -(B_3^T P_0 B_3 + D_{13}^T D_{13} + \varepsilon D_{23}^T D_{23})^{-1} \times \begin{bmatrix} B_3^T P_0 & D_{13}^T & \varepsilon D_{23}^T \end{bmatrix} \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix}.$$

Proof. By statement (6) in Proposition 2.7, we are interested in $J_1(G_\varepsilon^{cl})$ where G_ε^{cl} is the scaled closed loop system given by

$$G_\varepsilon^{cl} \sim \left[\begin{array}{c|cc} A & \sqrt{\varepsilon} B_1 & B_2 \\ \hline \frac{1}{\sqrt{\varepsilon}} C_1 & D_{11} & \frac{1}{\sqrt{\varepsilon}} D_{12} \\ C_2 & \sqrt{\varepsilon} D_{21} & D_{22} \end{array}\right] + \left[\begin{array}{c} B_3 \\ \hline \frac{1}{\sqrt{\varepsilon}} D_{13} \\ D_{23} \end{array}\right] \left[K_x \middle| \sqrt{\varepsilon} K_d \ K_w \right].$$

Thus, we equivalently reformulate the optimization problem as $\inf_{K_x, K_d, K_w} J_1(G_\varepsilon^{cl})$. Define

$$\bar{\phi} := (A,\ B_{[1,3]} T,\ \varepsilon^{-1} Q_\varepsilon,\ \varepsilon^{-1} T R_\varepsilon T,\ \varepsilon^{-1} S_\varepsilon T)$$
$$\bar{\omega} := (B_2,\ B_{[1,3]} T,\ \varepsilon^{-1} \bar{Q}_\varepsilon,\ \varepsilon^{-1} T R_\varepsilon T,\ \varepsilon^{-1} \bar{S}_\varepsilon T)$$

where $T := \operatorname{diag}(\sqrt{\varepsilon} I, I)$. Note that (A2) is equivalent for $J_1(G_\varepsilon^{cl})$ and $J_{(\varepsilon^{-1})}(G_1^{cl})$ because invertibility of $D_{13}^T D_{13} + D_{23}^T D_{23}$ is equivalent to invertibility of $\varepsilon^{-1} D_{13}^T D_{13} + D_{23}^T D_{23}$. Therefore, by Theorem 4.3, if the DARE $\mathcal{R}_{\bar{\phi}}(P) = P$ has a stabilizing solution $\bar{P}_0 \succeq 0$ such that $\mathcal{N}(T B_{[1,3]}^T \bar{P}_0 B_{[1,3]} T + \varepsilon^{-1} T R_\varepsilon T) = (n_u, 0, n_d)$, then $\inf_{K_x, K_d, K_w} J_1(G_\varepsilon^{cl}) = \operatorname{tr}\{\mathcal{R}_{\bar{\omega}}(\bar{P}_0)\}$. It is straightforward to show that $\mathcal{R}_{\bar{\phi}}(\varepsilon^{-1} P_0) = \varepsilon^{-1} \mathcal{R}_{\phi_\varepsilon}(P_0) = \varepsilon^{-1} P_0$ and $\mathcal{A}_{\bar{\phi}}(\varepsilon^{-1} P_0) = \mathcal{A}_{\phi_\varepsilon}(P_0)$. Therefore $\varepsilon^{-1} P_0 \succeq 0$ is the stabilizing solution of the DARE $\mathcal{R}_{\bar{\phi}}(P) = P$. Also note that, since $\mathcal{N}(B_{[1,3]}^T P_0 B_{[1,3]} + R_\varepsilon) = \mathcal{N}(\varepsilon^{-1} T (B_{[1,3]}^T P_0 B_{[1,3]} + R_\varepsilon) T)$, we have that $\inf_{K_x, K_d, K_w} J_1(G_\varepsilon^{cl}) = \operatorname{tr}\{\mathcal{R}_{\bar{\omega}}(\varepsilon^{-1} P_0)\}$. Since $\mathcal{R}_{\bar{\omega}}(\varepsilon^{-1} P_0) = \varepsilon^{-1} \mathcal{R}_{\omega_\varepsilon}(P_0)$, we have established the optimal cost of the optimization. Returning to Theorem 4.3, since $\bar{P}_0 = \varepsilon^{-1} P_0$, we see after some algebra that this performance is achieved by the scaled controller $[K_x \ \sqrt{\varepsilon} K_d \ K_w] = [K_{x,\varepsilon} \ \sqrt{\varepsilon} K_{d,\varepsilon} \ K_{w,\varepsilon}]$. □

In the preceding theorem, we assumed the existence of a DARE solution with several relevant properties. The next theorem gives a set of conditions which guarantee that the DARE solution has a solution with the required properties.

Theorem 4.6. *Let $\varepsilon > 0$ and suppose that (A1)–(A4) hold. Then $\exists K \in \mathcal{K}$ such that $J_{(\varepsilon^{-1})}(\mathcal{F}_l(G_{FI}, K)) \neq \infty \Leftrightarrow$ the DARE $\mathcal{R}_{\phi_\varepsilon}(P) = P$ has a stabilizing solution $P_0 \succeq 0$ such that $\mathcal{N}(B_{[1,3]}^T P_0 B_{[1,3]} + R_\varepsilon) = (n_u, 0, n_d)$.*

The proof of this theorem, although omitted for brevity, it straightforward upon noticing that performing the scaling in the proof of Theorem 4.5 does not change (A1)–(A4); we just apply Theorem 4.4 to the scaled system.

Now we present a result which makes it especially easy to find values of ε for which the optimal $J_{(\varepsilon^{-1})}$ is finite.

Theorem 4.7. *Let G_{FI} be given by Eq.* (3) *and denote $G^{cl}(K) := \mathcal{F}_l(G_{FI}, K)$. If $\inf_{K \in \mathcal{K}} J_{(\varepsilon^{-1})}(G^{cl}(K)) \neq \infty$, then $\inf_{K \in \mathcal{K}} J_{((\alpha\varepsilon)^{-1})}(G^{cl}(K)) \neq \infty$, $\forall \alpha \in (0,1]$.*

Proof. If $J_{(\varepsilon^{-1})}(G^{cl}(K)) \neq \infty$, then, by statement 3 of Proposition 2.7, $J_{((\alpha\varepsilon)^{-1})}(G^{cl}(K)) \neq \infty$, $\forall \alpha \in (0,1]$. Therefore, $\inf_{K \in \mathcal{K}} J_{((\alpha\varepsilon)^{-1})}(G^{cl}(K)) \neq \infty$, $\forall \alpha \geq 1$. □

We now analyze how the optimal cost varies as ε is varied. As in [1], we do so by taking derivatives of all relevant quantities with respect to ε. To simplify the following discussion, we will denote the optimal value of $J_{(\varepsilon^{-1})}(\mathcal{F}_l(G_{FI}, K))$ as J^o. We will also denote the optimal value of ε (assuming it exists) as ε^o. Defining

$$\hat{C} := C_2 - \begin{bmatrix} D_{21} & D_{23} \end{bmatrix} \left(B_{[1,3]}^T P_0 B_{[1,3]} + R_\varepsilon \right)^{-1} \left(B_{[1,3]}^T P_0 A + S_\varepsilon \right)$$
$$\hat{D} := D_{22} - \begin{bmatrix} D_{21} & D_{23} \end{bmatrix} \left(B_{[1,3]}^T P_0 B_{[1,3]} + R_\varepsilon \right)^{-1} \left(B_{[1,3]}^T P_0 B_2 + \bar{S}_\varepsilon \right)$$

the relevant derivatives are given by

$$P_0' = A_{\phi_\varepsilon}(P_0)^T P_0' A_{\phi_\varepsilon}(P_0) + \hat{C}^T \hat{C} \tag{11a}$$
$$\mathcal{R}_{\omega_\varepsilon}(P_0)' = A_{\omega_\varepsilon}(P_0)^T P_0' A_{\omega_\varepsilon}(P_0) + \hat{D}^T \hat{D} \tag{11b}$$
$$(J^o)' = \varepsilon^{-1}(\operatorname{tr}\{\mathcal{R}_{\omega_\varepsilon}(P_0)'\} - J^o). \tag{11c}$$

(Equation (11a) is obtained by implicitly differentiating the DARE $\mathcal{R}_{\phi_\varepsilon}(P_0) = P_0$.) It should be noted that, due to the stability of $A_{\phi_\varepsilon}(P_0)$, we can always solve the discrete Lyapunov equation

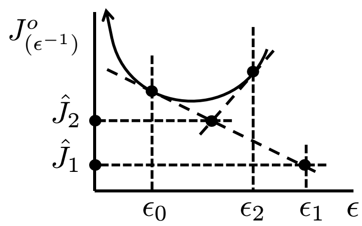


Figure 5: ILLUSTRATION OF LOWER BOUND COMPUTATION

in Eq. (11a) for P_0' once we have determined P_0 for a particular value of ε. For the best numerical properties, we should do so by solving for its Cholesky factor (e.g. using `dlyapchol` in MATLAB). We then use the Cholesky factor to evaluate $\mathcal{R}_{\omega_\varepsilon}(P_0)'$ using Eq. (11b). It should be noted that exploiting this structure guarantees that $\mathcal{R}_{\omega_\varepsilon}(P_0)' \succeq 0$ even in the face of numerical errors. Finally, the derivative of the optimal cost is given by Eq. (11c). It should be noted that if J^o exists and $(J^o)' < 0$ for a particular value of ε, then $\varepsilon < \varepsilon^o$. If these conditions do not hold, then $\varepsilon \geq \varepsilon^o$.

The value and derivative of J^o is also useful for generating a lower bound on the optimal value of Eq. (6). Consider Fig. 5, which shows a representative graph of J^o in which ε_0 is known to be an lower bound on ε^o. By convexity, if ε_1 is known to be a upper bound on ε^o, the value and derivative of J^o at ε_0 gives us the lower bound $\hat{J}_1$. If instead, the value and derivative of J^o at ε_2 are known, we have the lower bound $\hat{J}_2$. It should be noted that the second of these lower bounds is less conservative when it is applicable.

With these results in place, we can easily solve Eq. (6) using the following methodology:

Step 1—Find Initial Interval: Choose $\alpha > 1$. Check whether or not $1 < \varepsilon^o$. If so, start from $k = 1$ and increment k until $\alpha^k \geq \varepsilon^o$. Denoting this upper bound as ε_u, we see that $\varepsilon^o \in (\alpha^{-1}\varepsilon_u, \varepsilon_u)$. If instead $1 \geq \varepsilon^o$, start from $k = 1$ and increment k until $\alpha^{-k} < \varepsilon^o$. Denoting this lower bound as ε_l, we see that the optimal value of ε lies in the interval $(\varepsilon_l, \alpha\varepsilon_l)$.
Step 2—Bisection: Use bisection to find the optimal value of ε.

In our implementation, we use $\alpha = 100$ and, in the bisection step, we use the geometric mean instead of the arithmetic mean. We use two stopping criteria—defining the relative error as $\nu := 1 - \underline{f}/\mathrm{tr}\{W_0\}$ where $\underline{f}$ is the lower bound depicted in Fig. 5, we terminate the algorithm when either $\nu < 10^{-10}$ or the number of iterations (for both steps combined) exceeds 30.

5 NUMERICAL EXPERIMENTS

In this section, we consider the application of the developed methodologies to randomly generated FI $\mathcal{H}_2$ guaranteed cost control problems. In particular, we consider three approaches—using the DARE approach outlined in §4, solving Eq. (4) using

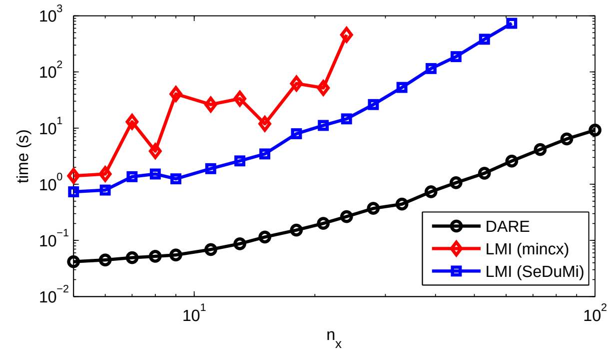


Figure 6: TIME REQUIRED TO SOLVE RANDOMLY GENERATED FI H2 GUARANTEED COST CONTROL PROBLEMS

SeDuMi (parsed using YALMIP [11]), and solving Eq. (4) using the `mincx` command in the Robust Control Toolbox. The last two of these methods will be collectively called the LMI methods. It should be noted that YALMIP was not used when using `mincx` because YALMIP causes `mincx` to run more slowly. All numerical experiments were performed in MATLAB (with multithreaded computation disabled) on a computer with a 2.2 GHz Intel Core 2 Duo Processor and 2 GB of RAM.

To generate the random systems in our numerical experiments, we first generated a random stable discrete-time state space system using `drss` in MATLAB, designed an optimal (non-robust) FI $\mathcal{H}_2$ controller, and then multiplied the closed loop system by the inverse of its $\mathcal{H}_\infty$ norm (computed by `norm`). This system was then multiplied by a random number generated from a uniform distribution on $[-1, 1]$. The resulting system corresponded to generating random values of A, B_1, C_1, and D_{11} for a robustly stable system. The FI $\mathcal{H}_2$ control step was used as a heuristic to make the control design problems less well-conditioned. (In particular, this tends to result in systems which are "closer" to not being robustly stabilizable.) We then generated random values of $B_2, B_3, C_2, D_{12}, D_{13}, D_{21}, D_{22}, D_{23}, K_x$, and K_d from independent normal distributions. Finally, we set $A \leftarrow A + B_3K_x$, $B_1 \leftarrow B_1 + B_3K_d$, $C_1 \leftarrow C_1 + D_{13}K_x$, and $D_{11} \leftarrow D_{11} + D_{13}K_d$. Note that this corresponds to "shifting" the system by a randomly chosen control scheme; although the resulting system is not guaranteed to be stable, it is guaranteed that an FI control scheme exists which robustly stabilizes the system. For all of the numerical experiments, we chose the signal dimensions to be $n_q = 8, n_p = 7, n_d = 6, n_w = 5, n_u = 4$.

In the first experiment, we tested the speed of the methodologies over several values of n_x, the dimension of the plant state. The results of this test are shown in Fig. 6. In particular, note that the DARE method is faster than the LMI methods for all of the randomly generated problems. For instance, for the 24^{th}-order system, it respectively took the DARE approach, the `mincx` approach, and the SeDuMi approach 0.27 seconds, 14.59 seconds, and 457.60 seconds to compute the opti-

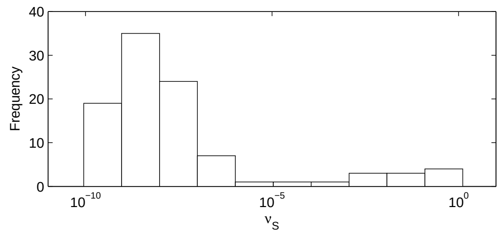


Figure 7: HISTOGRAM OF $\nu_s > 0$

mal achievable performance and construct the optimal controller. Also note that the DARE method appears to have a complexity of $O(n_x^3)$ whereas the SeDumi method appears to have a complexity of $O(n_x^4)$. The curve which corresponds to the `mincx` method is not smooth because the number of iterations required to solve the problem often changes dramatically from problem to problem, unlike the other two methods. Nonetheless, solving the problem using `mincx` appears to have a complexity of at least $O(n_x^4)$ also. Thus, the difference in computational speed between the DARE approach and the other two approaches becomes more pronounced for larger values of n_x.

In the second experiment, we tested the accuracy of the DARE approach compared to the LMI approaches for 100 randomly generated analysis problems with $n_x = 20$. To this end, we first define f_d, f_m, and f_s as the $\mathcal{H}_2$ guaranteed cost performance (determined using the algorithm of [1]) for the optimal closed loop systems respectively computed using the DARE approach, `mincx`, and SeDuMi. The criterion we will be using to compare the accuracy of the relevant methods is the relative error, i.e. we use the criterion $\nu_m := f_m/f_d - 1$ to compare the accuracy of the `mincx` approach to the DARE approach and the criterion $\nu_s := f_s/f_d - 1$ to compare the accuracy of the SeDuMi approach to the DARE approach. For 98 of the random systems used in this paper, $|\nu_m| < 10^{-10}$, i.e. `mincx` almost always achieved comparable accuracy to the DARE approach. For the other two systems, the relative error was 0.03% and 4%. (Note that these two cases correspond to `mincx` getting "stuck" in a suboptimal solution.) SeDuMi, however, tended not to be quite as accurate. As shown in Fig. 7, the SeDuMi method frequently gets "stuck" in a suboptimal solution due to numerical problems. For one of the random systems, SeDuMi returned a controller which did not robustly stabilize the system. Also, for 99 of the random systems used in this paper, $\nu_s > 0$ (i.e. SeDuMi achieved inferior accuracy). For the one remaining system, $\nu_s = -13\%$. However, upon closer examination, we found that this was a numerical error arising in the *analysis* algorithm, not the synthesis algorithm developed in this paper. When we made a small perturbation on the initial condition for the analysis algorithm, it certified that both closed loop systems achieved comparable performance. Thus, we conclude that the accuracy of the DARE method is equivalent or superior to that of the LMI methods.

6 CONCLUSION

In this paper, we formulated the problem of finding the best full information $\mathcal{H}_2$ guaranteed cost controller of a discrete-time system with dynamic norm-bounded unstructured uncertainty as a SDP. We then demonstrated that exploiting the structure of this optimization by using the solution of DAREs increases the speed and accuracy with which we can solve these problems.

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